

Asymptotic expansion of β matrix models in the one-cut regime

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Abstract

We prove the existence of a $1/N$ expansion to all orders in β matrix models with a confining, offcritical potential corresponding to an equilibrium measure with a connected support. Thus, the coefficients of the expansion can be obtained recursively by the "topological recursion" derived in [CE06]. Our method relies on the combination of a priori bounds on the correlators and the study of Schwinger-Dyson equations, thanks to the uses of classical complex analysis techniques. These a priori bounds can be derived following [dMPS95, Joh98, KS10] or for strictly convex potentials by using concentration of measure [AGZ10, Section 2.3]. Doing so, we extend the strategy of [GMS07], from the hermitian models ($\beta = 2$) and perturbative potentials, to general β models. The existence of the first correction in $1/N$ was considered in [Joh98] and more recently in [KS10]. Here, by taking similar hypotheses, we extend the result to all orders in $1/N$.

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1 Introduction

1.1 Definitions

We consider the probability measure $\mu_{N,\beta}^V$ on \mathbb{R}^N given by:

$$d\mu_{N,\beta}^{V;[b_-,b_+]}(\lambda) = \frac{1}{Z_{N,\beta}^{V;[b_-,b_+]}} \prod_{i=1}^N d\lambda_i e^{-\frac{N\beta}{2} V(\lambda_i)} \mathbf{1}_{[b_-,b_+]}(\lambda_i) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta. \quad (1-1)$$

$[b_-, b_+]$ is an interval of the real line, $-\infty \leq b_- < b_+ \leq +\infty$, and β is a positive number. For $\beta = 1, 2, 4$, this is the measure induced on the eigenvalues of Φ by the probability measure $d\Phi e^{-\frac{N\beta}{2} \text{Tr} V(\Phi)}$ on a vector space $\mathcal{E}_{N,\beta}$ of $N \times N$ matrices. $\mathcal{E}_{N,1}$ (resp. $\mathcal{E}_{N,2}$, and $\mathcal{E}_{N,4}$) is the space of real symmetric (resp. hermitian, and quaternionic self-dual) matrices [Meh04]. For general $\beta > 0$, when V is quadratic (Hermite weight), or log + linear (Laguerre weight), Dumitriu and Edelman have found [DE02] a measure on a set of tridiagonal matrices which induces the measure $d\mu_{N,\beta}^{V;\mathbb{R}}$ on eigenvalues. This can be generalized to any even polynomial potential [Rid]. This was subsequently exploited to study these particular β matrix models by Ramírez, Rider and Virág [RRV06]. Though, there is no known plain random matrix whose spectrum is distributed according to $d\mu_{N,\beta}^{V;\mathbb{R}}$ for general β and V , we still speak of a "matrix model", and we call λ_i the eigenvalues.

We define the unnormalized empirical measure M_N of the eigenvalues given by

$$M_N = \sum_{i=1}^N \delta_{\lambda_i}$$

and their Cauchy-Stieltjes transform, which are generating series for the moments of M_N . In fact, we prefer to work with the generating series of the cumulants. They are also called "correlators", and are defined for $x_1, \dots, x_n \in \mathbb{C} \setminus [b_-, b_+]$ by

$$\begin{aligned} W_n^{V;[b_-,b_+]}(x_1, \dots, x_n) &= \mu_{N,\beta}^{V;[b_-,b_+]} \left[\left(\int \frac{dM_N(\xi_1)}{x_1 - \xi_1} \dots \int \frac{dM_N(\xi_n)}{x_n - \xi_n} \right)_c \right] \\ &= \partial_{\epsilon_1} \dots \partial_{\epsilon_n} \left(\ln Z_{N,\beta}^{V - \frac{2}{\beta N} \sum_i \frac{\epsilon_i}{x_i - \bullet}; [b_-, b_+]} \right) \Big|_{\epsilon_i=0}. \end{aligned}$$

In particular, we have

$$\begin{aligned} W_1^{V;[b_-,b_+]}(x) &= \mu_{N,\beta}^{V;[b_-,b_+]} \left[\int \frac{dM_N(\xi)}{x - \xi} \right], \\ W_2^{V;[b_-,b_+]}(x_1, x_2) &= \mu_{N,\beta}^{V;[b_-,b_+]} \left[\iint \frac{dM_N(\xi)}{x_1 - \xi} \frac{dM_N(\eta)}{x_2 - \eta} \right] \\ &\quad - \mu_{N,\beta}^{V;[b_-,b_+]} \left[\int \frac{dM_N(\xi)}{x_1 - \xi} \right] \mu_{N,\beta}^{V;[b_-,b_+]} \left[\int \frac{dM_N(\eta)}{x_2 - \eta} \right]. \end{aligned}$$

When there is no confusion, we may omit to write the dependence in V and $[b_-, b_+]$ in the exponent.

It is well-known, see [Joh98] or the textbooks [Dei99, Theorem6] or [AGZ10, Theorem 2.6.1 and Corollary 2.6.3], that

Theorem 1.1 *Assume that $V : [b_-, b_+] \rightarrow \mathbb{R}$ is a continuous function, and if $b_\tau = \tau\infty$ is infinite, assume that:*

$$\liminf_{x \rightarrow \tau\infty} \frac{V(x)}{2 \ln |x|} > 1.$$

If V depends on N , assume also that $V \rightarrow V^{\{0\}}$ in the space of continuous function over $[b_-, b_+]$ for the sup norm. Then, the normalized empirical measure $L_N = N^{-1} M_N$ converges almost surely and in expectation towards the unique probability measure $\mu_{\text{eq}} := \mu_{\text{eq}}^{V; [b_-, b_+]}$ on $[b_-, b_+]$ which minimizes:

$$\mathcal{E}[\mu] = \int d\mu(\xi) V^{\{0\}}(\xi) - \iint d\mu(\xi) d\mu(\eta) \ln |\xi - \eta|.$$

Moreover, μ_{eq} has compact support. It is characterized by the existence of a constant C such that:

$$\begin{cases} \forall x \in [b_-, b_+] & 2 \int_{b_-}^{b_+} d\mu_{\text{eq}}(\xi) \ln |x - \xi| - V^{\{0\}}(x) \leq C, \\ \mu_{\text{eq}} \text{ almost surely} & 2 \int_{b_-}^{b_+} d\mu_{\text{eq}}(\xi) \ln |x - \xi| - V^{\{0\}}(x) = C. \end{cases} \quad (1-2)$$

In particular, for any $x \in \mathbb{C} \setminus [b_-, b_+]$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} W_1(x) = \int \frac{d\mu_{\text{eq}}(\xi)}{x - \xi} := W_1^{\{-1\}}(x),$$

and the convergence is uniform in any compact of $\mathbb{C} \setminus [b_-, b_+]$.

1.2 Main results

Our goal is to prove an asymptotic expansion in powers of $1/N$ when $N \rightarrow \infty$ for the partition function $Z_{N,\beta}^{V; [b_-, b_+]}$ and the correlators $W_n^{V; [b_-, b_+]}(x_1, \dots, x_n)$. This is not always expected. In particular it is false when the support of $\mu_{\text{eq}}^{V; [b_-, b_+]}$, the limiting eigenvalue distribution, is not connected: corrections to the leading order feature a quasi periodic behavior with N (see [Eyn09] for a general heuristic argument). Our proof uses a priori bounds on the correlators, and what we really need is to establish that $W_n \in O(1)$ for $n \geq 2$. We shall prove this condition either based on a result of Boutet de Monvel, Pastur and Shcherbina [dMPS95] (also used recently in the context of β ensembles by Kriecherbauer and Shcherbina [KS10]), or under the additional assumption that V is strictly convex. In the convex setting, such a priori bounds can be derived from concentration of measures properties, in which case our article is self-contained. Our basic assumptions and main results are:

Hypothesis 1.1

- (Regularity) $V : [b_-, b_+] \rightarrow \mathbb{R}$ is continuous, and if V depends on N , it has a limit $V^{\{0\}}$ in the space of continuous functions over $[b_-, b_+]$ for the sup norm.
- (Confinement) If $b_\tau = \tau\infty$, $\liminf_{x \rightarrow \tau\infty} \frac{V(x)}{2 \ln |x|} > 1$.
- (One-cut regime) The support of $\mu_{\text{eq}}^{V; [b_-, b_+]}$ consists in a unique interval $[\alpha_-, \alpha_+] \subseteq [b_-, b_+]$.
- (Control of large deviations) The function $x \in [b_-, b_+] \setminus]\alpha_-, \alpha_+[\mapsto \frac{1}{2}V(x) - \int \ln |x - \xi| d\mu_{\text{eq}}(\xi)$ achieves its minimum value at α_- and α_+ only.
- (Offcriticality) $S(x) > 0$ whenever $x \in [\alpha_-, \alpha_+]$, where:

$$S(x) = \pi \frac{d\mu_{\text{eq}}}{dx} \sqrt{\left| \frac{\prod_{\tau' \in \text{Hard}} (x - \alpha_{\tau'})}{\prod_{\tau \in \text{Soft}} (x - \alpha_\tau)} \right|},$$

where $\tau \in \text{Hard}$ (resp. $\tau \in \text{Soft}$) iff $b_\tau = \alpha_\tau$ (resp. $\tau(b_\tau - \alpha_\tau) > 0$).

- (Analyticity) V can be extended as a holomorphic function in some open neighborhood of $[\alpha_-, \alpha_+]$.
- V has a $1/N$ expansion in this neighborhood, in the sense of Hyp. 4.5.

Notice that the "one-cut regime", "offcriticality" and "control of large deviations" assumptions automatically hold when V is strictly convex (see [Joh98, Proposition 3.1], which extends easily to analytic functions instead of polynomials).

Proposition 1.1 *Assume Hyp. 1.1. Then, $W_n^{V; [b_-, b_+]}$ admits an asymptotic expansion when $N \rightarrow \infty$:*

$$W_n^{V; [b_-, b_+]}(x_1, \dots, x_n) = \sum_{k \geq n-2} N^{-k} W_n^{V; \{k\}}(x_1, \dots, x_n),$$

which has the precise meaning that, for all $K \geq n - 2$:

$$W_n^{V; [b_-, b_+]}(x_1, \dots, x_n) = \sum_{k=n-2}^K N^{-k} W_n^{V; \{k\}}(x_1, \dots, x_n) + o(N^{-K}).$$

The $o(N^{-K})$ is uniform for x_1, \dots, x_n in any compact of $(\mathbb{C} \setminus [b_-, b_+])^n$, but not uniform in n and K . Moreover, if $(b_\tau - \alpha_\tau)\tau > 0$ (meaning that α_τ is a soft edge) and Hypotheses 1.1 hold, the functions $W_n^{V; \{k\}}$ are independent of b_τ .

Proposition 1.2 *Assume Hyp. 1.1, and $b_- < \alpha_- < \alpha_+ < b_+$ (all edges are soft). Then, $Z_{N,\beta}^{V;[b_-,b_+]}$ admits an asymptotic expansion when $N \rightarrow \infty$:*

$$Z_{N,\beta}^{V;[b_-,b_+]} = Z_{N,G\beta E} \left(\frac{\alpha_+ - \alpha_-}{4} \right)^{N+\beta \frac{N(N-1)}{2}} \exp \left(\sum_{k \geq -2} N^{-k} F_\beta^{V;\{k\}} \right). \quad (1-3)$$

In other words, for any $K \geq -2$:

$$Z_{N,\beta}^{V;[b_-,b_+]} = Z_{N,G\beta E} \left(\frac{\alpha_+ - \alpha_-}{4} \right)^{N+\beta \frac{N(N-1)}{2}} \exp \left(\sum_{k=-2}^K N^{-k} F_\beta^{V;\{k\}} + o(N^{-K}) \right).$$

Moreover, the coefficients $F_\beta^{V;\{k\}}$ are independent of b_- and b_+ chosen such that $b_- < \alpha_- < \alpha_+ < b_+$ and Hypotheses 1.1 hold.

$Z_{N,G\beta E}$ is the partition function of the Gaussian β ensemble, defined by the quadratic potential $V_G(x) = \frac{x^2}{2}$. It is given by a Selberg integral [Sel44] (also in [Meh04]):

$$Z_{N,G\beta E} = (2\pi)^{N/2} (N\beta/2)^{-\beta N^2/4 + (\beta/4 - 1/2)N} \frac{\prod_{j=1}^N \Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)^N}. \quad (1-4)$$

For hard edges (i.e. $b_- = \alpha_-$ or $b_+ = \alpha_+$), one may still interpolate between $Z_{N,\beta}^{V;[b_-,b_+]}$ and a Gaussian β ensemble restricted to some interval (Corollary 5.1), but the partition function of the latter is not a Selberg integral and thus not known in closed form.

1.2.1 Commentary

When V does not depend on N and β , $W_n^{V;\{k\}}$ has a very simple dependence in β :

$$W_n^{V;\{k\}} = \sum_{g=0}^{\lfloor (k-n+2)/2 \rfloor} \left(\frac{\beta}{2} \right)^{1-g-n} \left(1 - \frac{2}{\beta} \right)^{k+2-2g-n} \mathcal{W}_n^{V;(g;k+2-2g-n)}, \quad (1-5)$$

and likewise:

$$F_\beta^{V;\{k\}} = \sum_{g=0}^{\lfloor k/2 \rfloor + 1} \left(\frac{\beta}{2} \right)^{1-g} \left(1 - \frac{2}{\beta} \right)^{k+2-2g} \mathcal{F}^{V;(g;k+2-2g)}. \quad (1-6)$$

Assuming existence of the $1/N$ expansion, or at the level of formal matrix integrals, the recursive computation of the $\mathcal{W}_n^{V;(g;l)}$ and $\mathcal{F}^{V;(g;l)}$ was developed by Chekhov and Eynard in [CE06]. For $\beta = 2$, it is well-known that Eqn. 1-3 is an expansion in even powers of N , i.e. $F_{\beta=2}^{V;\{2k+1\}} = 0$. Such a result goes back to the so-called topological expansion of t'Hooft, shown in the context of matrix models by Brézin, Itzykson,

Parisi and Zuber [BIPZ78]. Indeed when $\beta = 2$, the sum in Eqn. 1-5 has only one term, namely $k = 2g - 2 + n$, which is present only when $k = n \pmod{2}$, and likewise for Eqn. 1-6 which can be considered as the case $n = 0$.

At the asymptotic level, the case $\beta = 2$ was tackled in [APS01]. For $\beta = 1, 2, 4$, the partition function and the correlators can be computed with the help of orthogonal polynomials [Meh04]. These are solutions of a Riemann-Hilbert problem [FIK92], for which the large N asymptotics have been intensively studied [BI99, DKM⁺97, DKM⁺99b, DKM⁺99a] with the steepest descent method introduced in [DZ95]. As a consequence, Ercolani and McLaughlin [EM03] were able to prove the existence of a $1/N^2$ expansion of $\ln Z_{N,\beta=2}^V$. However, the topological expansions in the cases $\beta = 1$ and 4 are technically more involved in this framework, and have resisted to analysis up to now.

Integrability properties of β matrix models are unraveled for general $\beta > 0$, in particular there is no known orthogonal polynomials techniques to evaluate the partition function $Z_{N,\beta}^V$ and the correlation functions $W_n(x_1, \dots, x_n)$. Yet, it is always possible to study the Cauchy-Stieltjes transform of the empirical measure of the eigenvalues and the "loop equations", also called Schwinger-Dyson equations or Pastur equations [Pas72], that govern its expectations and cumulants. Thanks to the rough bounds for W_1^V and W_2^V established in [dMPS95], Johansson [Joh98] proved a central limit theorem and obtained the first correction to W_1 when V is an even polynomial satisfying Hyp. 1.1. This was also the subject of a recent work by Kriecherbauer and Shcherbina [KS10], with Hyp. 1.1 only. These authors have obtained in particular the expansion of $\ln Z_{N,\beta}^V$ up to a $O(1)$ when $N \rightarrow \infty$ (see their Theorem 2).

The determination of $W_1^{V;\{-1\}}$ [Wig58, BIPZ78, AG97] and $W_2^{V;\{0\}}$ [AM90, Bee94, Joh98] has been known for long, in β ensembles or many other matrix models. It was also observed long ago [AM90] that, if a $1/N$ expansion is assumed to exist, the loop equations turn into a system of recursive linear equations determining fully the decaying orders. To solve it, one just has to invert a linear operator \mathcal{K} . Recursiveness is a consequence of the assumption or the fact that $W_n \in O(N^{2-n})$, which allows the determination of the leading order of W_{n-1} without knowledge of W_n (for $n \geq 3$). These techniques found their origin in [AM90, ACM92, ACKM93, ACKM95] and culminated with the formalism of the "topological recursion" of [Eyn04, EO07] for $\beta = 2$, and [CE06] for any fixed $\beta > 0$.

In this article, we observe that \mathcal{K}^{-1} is a continuous operator on some appropriate space of analytic functions. Combining with the a priori control on correlators which dates back to [dMPS95], we prove the existence of the full expansion.

For strictly convex potentials, concentration inequalities also provide rough bounds

on the correlators and therefore allow us to give self-contained proofs, independent from [dMPS95]-[KS10]. In this framework, loop equations were used in [GMS07] to establish the asymptotic expansion of models of several hermitian random matrices ($\beta = 2$) with strictly convex interactions. Maurel-Segala [MS] also studied models of several symmetric random matrices ($\beta = 1$) with strictly convex interactions. In order to prove the asymptotic expansion, the main step of [GMS07] was to show that some operator on non-commutative polynomials could be inverted, with bounded appropriate norm, and this was only done in a perturbative regime. Here, thanks to complex analysis, the potential need not be a small perturbation of the quadratic potential.

Our techniques could also be applied to other matrix models. For instance, the convergent β , $\mathcal{O}(\mathbf{n})$ matrix model:

$$d\mu_{N,\beta,\mathcal{O}(\mathbf{n})}^{V;\mathbb{R}_+}(\lambda) = \frac{1}{Z_{N,\beta,\mathcal{O}(\mathbf{n})}^{V;\mathbb{R}_+}} \prod_{i=1}^N d\lambda_i e^{-\frac{N\beta}{2} V(\lambda_i)} \frac{\prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta}{\prod_{1 \leq i, j \leq N} (\lambda_i + \lambda_j)^{\mathbf{n}/2}}.$$

An important point is that the corresponding quadratic functional:

$$\mathcal{E}[\rho] = \iint_{\mathbb{R}_+^2} d\rho(\xi) d\rho(\eta) \left[-\frac{\beta}{2} \ln |\xi - \eta| + \frac{\mathbf{n}}{2} \ln |\xi + \eta| \right] + \frac{\beta}{2} \int_{\mathbb{R}_+} d\rho(\xi) V^{\{0\}}(\xi)$$

is strictly convex in the regime $|\mathbf{n}| < \beta$, therefore ensuring uniqueness of its minimizer. Besides, the analytic tools for the recursive determination of the one-cut solution to the loop equations of the $\mathcal{O}(\mathbf{n})$ model in this regime were clarified in [BE11]. The existence of a full $1/N$ expansion for convergent $\mathcal{O}(\mathbf{n})$ matrix models under Hyp. 1.1 could probably be established by following the lines we are presenting for the β matrix models.

An open challenge, which would be interesting for further applications, is to extend the same reasoning to complex measures, i.e. to Eqn. 1-1 where the potential V is complex-valued, or/and where the eigenvalues are integrated on a given path in the complex plane.

Outline of the article

We first study in Section 2 the weak dependence in the bounds of integration under weak assumptions on V . In particular, we may trade the initial interval $[b_-, b_+]$ for a finite interval $[a_-, a_+]$. We then write in Section 3 the corresponding loop equations for the correlators. Section 4 is devoted to the proof of the asymptotic expansion of the correlators with slightly stronger assumptions (Prop. 4.1). They are weakened in Section 5 to complete the proof of our main results for the correlators (Prop. 1.1) and the free energy (Prop. 1.2). We also remind how early steps of our proof imply the central limit theorem of Johansson (Prop. 5.2).

2 Weak dependence on the soft edges

In this section we show that the partition function and the correlators depend very weakly on the boundary points of the interval of integration $[b_-, b_+]$ if they are soft, i.e. do not coincide with the boundary points of the support $[\alpha_-, \alpha_+]$ of the equilibrium measure. We show more precisely that this dependence yields only exponentially small corrections, by deriving a large deviation principle for the law of the extreme eigenvalues. This point was already studied in [AGZ10, section 2.6.2] under a technical assumption [AGZ10, Assumption 2.6.5] that we replace here by assuming that the rate function of our large deviation principle vanishes only at α_- and α_+ . This result is not new in essence and not specific to the one-cut regime, see for instance [APS01, Proposition 2] where it is proved with the extra assumption that V has bounded second derivatives in a neighborhood of $\text{supp } \mu^{\text{eq}}$, or [PS11, Proposition 11.1.4] where it is proved with the extra assumption that V satisfies a Lipschitz condition in $[b_-, b_+]$.

2.1 Large deviation principle for the extreme eigenvalues

Under the assumptions of Theorem 1.1 on an interval $[b_-, b_+]$, we can define:

$$\mathcal{J}^{V;[b_-, b_+]}(x) = \frac{V(x)}{2} - \int_{b_-}^{b_+} d\mu_{\text{eq}}^{V;[b_-, b_+]}(\xi) \ln |x - \xi|$$

when $x \in [b_-, b_+]$, and $+\infty$ otherwise. Suppose that $[b_-, b_+] \neq [\alpha_-, \alpha_+]$, and set:

$$\tilde{\mathcal{J}}^{V;[b_-, b_+]}(x) = \mathcal{J}^{V;[b_-, b_+]}(x) - \inf_{\xi \in [b_-, b_+]} \mathcal{J}^{V;[b_-, b_+]}(\xi).$$

We define also $\tilde{\mathcal{J}}_{\max}^{V;[b_-, b_+]}(x)$ (resp. $\tilde{\mathcal{J}}_{\min}^{V;[b_-, b_+]}(x)$) which is equal to $\tilde{\mathcal{J}}^{V;[b_-, b_+]}(x)$, except when $x \in]-\infty, \alpha_-]$ (resp. $[\alpha_+, +\infty[$) where we set its value to $+\infty$.

Proposition 2.1 *Let $V : [b_-, b_+] \rightarrow \mathbb{R}$ be a continuous function, and if $b_\tau = \tau\infty$, assume that:*

$$\liminf_{x \rightarrow \tau\infty} \frac{V(x)}{2 \ln |x|} > 1. \quad (2-1)$$

Assume that $\tilde{\mathcal{J}}^{V;[b_-, b_+]}$ does not vanish outside $[\alpha_-, \alpha_+]$. Then:

- (i) $\beta \tilde{\mathcal{J}}_{\max}^{V;[b_-, b_+]}$ (resp. $\beta \tilde{\mathcal{J}}_{\min}^{V;[b_-, b_+]}$) is a good rate function on $[b_-, b_+]$, which vanishes at α_+ (resp. α_-).
- (ii) The law of λ_{\max} (resp. λ_{\min}) under $\mu_{N,\beta}^{V;[b_-, b_+]}$ satisfies a large deviation principle with speed N and rate function equal to $\beta \tilde{\mathcal{J}}_{\max}^{V;[b_-, b_+]}$ (resp. $\beta \tilde{\mathcal{J}}_{\min}^{V;[b_-, b_+]}$) on $[b_-, b_+]$.

In other words, for any closed subset F , or open subset Ω , of $[b_-, b_+]$:

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;[b_-,b_+]} (\lambda_{\max} \in F) &\leq -\beta \inf_{x \in F} \tilde{\mathcal{J}}_{\max}^{V;[b_-,b_+]}(x), \\ \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;[b_-,b_+]} (\lambda_{\max} \in \Omega) &\geq -\beta \inf_{x \in \Omega} \tilde{\mathcal{J}}_{\max}^{V;[b_-,b_+]}(x), \end{aligned}$$

and similar statements hold for λ_{\min} .

In particular, for any $\varepsilon > 0$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;[b_-,b_+]} (\lambda_{\min} \leq \alpha_- - \varepsilon) < 0, \quad (2-2)$$

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;[b_-,b_+]} (\lambda_{\max} \geq \alpha_+ + \varepsilon) < 0. \quad (2-3)$$

We give a proof of this proposition for completeness in the Appendix.

2.2 Weak dependence on the soft edges

We first state the global version of the result:

Proposition 2.2 *Let $V : [b_-, b_+] \rightarrow \mathbb{R}$ be a continuous function, and if $b_\tau = \tau\infty$, assume (2-1). Suppose $b_- < \alpha_-$, and assume furthermore that the minimum value of $\mathcal{J}^{V;[b_-,b_+]}$ is achieved only on $[\alpha_-, \alpha_+]$. Then, for any $\varepsilon > 0$, there exists $\eta_\varepsilon > 0$ so that:*

$$Z_{N,\beta}^{V;[b_-,b_+]} = Z_{N,\beta}^{V;[\alpha_- - \varepsilon, b_+]} (1 + O(e^{-N\eta_\varepsilon})),$$

and there exists a universal constant $\gamma_n > 0$ such that, for any $x_1, \dots, x_n \in (\mathbb{C} \setminus [b_-, b_+])^n$:

$$|W_n^{V;[b_-,b_+]}(x_1, \dots, x_n) - W_n^{V;[\alpha_- - \varepsilon, b_+]}(x_1, \dots, x_n)| \leq \frac{\gamma_n e^{-N\eta_\varepsilon}}{\prod_{i=1}^n d(x_i, [b_-, b_+])}. \quad (2-4)$$

A similar result holds for the upper edge.

We also have a local version:

Proposition 2.3 *Let $V : [b_-, b_+] \rightarrow \mathbb{R}$ be a continuous function, and if $b_\tau = \tau\infty$, assume (2-1). Suppose $b_- < \alpha_+$, and assume furthermore that the minimum value of $\mathcal{J}^{V;[b_-,b_+]}$ is achieved only on $[\alpha_-, \alpha_+]$. For any $\varepsilon > 0$ small enough, there exists $\eta_\varepsilon > 0$ so that, for any $a_- \in]b_-, \alpha_- - \varepsilon[$:*

$$\left| \partial_{a_-} \ln Z_N^{V;[a_-, b_+]} \right| \leq e^{-N\eta_\varepsilon},$$

and, for any $x_1, \dots, x_n \in (\mathbb{C} \setminus [a_-, b_+])^n$:

$$\forall x_1, \dots, x_n \in \mathbb{C} \setminus [a_-, b_+], \quad \left| \partial_{a_-} W_n^{V;[a_-, b_+]}(x_1, \dots, x_n) \right| \leq \frac{\gamma_n N^n}{\prod_{i=1}^n d(x_i, [a_-, b_+])} e^{-N\eta_\varepsilon}.$$

A similar statement holds for derivatives with respect to the upper bound.

Proof. If $b_- \neq \alpha_-$, let $a_- \in]b_-, \alpha_-[$. Notice that:

$$\left(1 - \frac{Z_{N,\beta}^{V;[a_-,b_+]}}{Z_{N,\beta}^{V;[b_-,b_+]}}\right) = \mu_{N,\beta}^{V;[b_-,b_+]}\left[\lambda_{\min} \leq a_-\right]. \quad (2-5)$$

If now $\phi : [b_-, b_+]^N \rightarrow \mathbb{C}$ is a bounded continuous function, we can write:

$$\mu_{N,\beta}^{V;[b_-,b_+]}\left[\phi(\lambda)\right] - \mu_{N,\beta}^{V;[a_-,b_+]}\left[\phi(\lambda)\right] = \mu_{N,\beta}^{V;[b_-,b_+]}\left[\phi(\lambda) \mathbf{1}_{\lambda_{\min} \leq a_-}\right] + \left(\frac{Z_{N,\beta}^{V;[a_-,b_+]}}{Z_{N,\beta}^{V;[b_-,b_+]}} - 1\right) \mu_{N,\beta}^{V;[a_-,b_+]}\left[\phi(\lambda)\right].$$

Thus, we find:

$$\left|\mu_{N,\beta}^{V;[b_-,b_+]}\left[\phi(\lambda)\right] - \mu_{N,\beta}^{V;[a_-,b_+]}\left[\phi(\lambda)\right]\right| \leq 2 \left(\sup_{\lambda \in [b_-, b_+]^N} |\phi(\lambda)|\right) \mu_{N,\beta}^{V;[b_-,b_+]}\left[\lambda_{\min} \leq a_-\right].$$

This can be applied for the disconnected correlators:

$$\overline{W}_n^{V;[b_-,b_+]}\left(x_1, \dots, x_n\right) = \mu_{N,\beta}^{V;[b_-,b_+]}\left[\prod_{j=1}^n \sum_{i_j=1}^N \frac{1}{x_j - \lambda_{i_j}}\right],$$

and we obtain:

$$\left|\overline{W}_n^{V;[b_-,b_+]}\left(x_1, \dots, x_n\right) - \overline{W}_n^{V;[a_-,b_+]}\left(x_1, \dots, x_n\right)\right| \leq \frac{2 N^n}{\prod_{j=1}^n d(x_j, [b_-, b_+])} \mu_{N,\beta}^{V;[b_-,b_+]}\left[\lambda_{\min} \leq a_-\right].$$

Similarly, one finds:

$$\left|\partial_{a_-} W_n^{V;[a_-,b_+]}\left(x_1, \dots, x_n\right)\right| \leq \frac{2 N^n}{\prod_{j=1}^n d(x_j, [a_-, b_+])} \partial_{a_-} \ln Z_{N,\beta}^{V;[a_-,b_+]}. \quad (2-6)$$

The correlators $W_n^{V;[b_-,b_+]}$ are just sums of monomials of the form $W_{n_1}^{V;[b_-,b_+]}\left(I_1\right) \dots W_{n_m}^{V;[b_-,b_+]}\left(I_m\right)$ where I_1, \dots, I_m is a partition of $\{x_1, \dots, x_n\}$.

So, it is enough to establish the weak dependence at the level of the partition function.

The global version is a direct consequence of Eqn. 2-2 applied to Eqn. 2-5:

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \left(1 - \frac{Z_{N,\beta}^{V;[a_-,b_+]}}{Z_{N,\beta}^{V;[b_-,b_+]}}\right) < 0.$$

For the local version, we rather need to bound:

$$\partial_{a_-} \ln Z_{N,\beta}^{V;[a_-,b_+]} = N \frac{Z_{N-1,\beta}^{\frac{NV}{N-1};[a_-,b_+]}}{Z_{N,\beta}^{V;[a_-,b_+]}} \mu_{N-1,\beta}^{\frac{NV}{N-1};[a_-,b_+]}\left[e^{\beta\left(-\frac{NV(a_-)}{2} + \sum_{i=1}^{N-1} \ln |\lambda_i - a_-|\right)}\right].$$

If $a_- \in]b_-, \alpha_-[$ is fixed, by the large deviation principle for L_{N-1} under $\mu_{N-1}^{\frac{NV}{N-1};[a_-,b_+]}$, since the logarithm is a lower semicontinuous function, there exists $\gamma > 0$ such that, for any $\epsilon > 0$, for N large enough:

$$\mu_{N-1,\beta}^{\frac{NV}{N-1};[a_-,b_+]}\left[e^{\beta\left(-\frac{NV(a_-)}{2} + \sum_{j=1}^{N-1} \ln |a_- - \lambda_j|\right)}\right] \leq \gamma e^{-\beta N(1-\epsilon)\mathcal{J}^{V;[b_-,b_+]}\left(a_-\right)}.$$

Moreover, we have seen in Eqn. 1-4 that for N large enough:

$$\frac{Z_{N-1,\beta}^{NV;[a_-,b_+]}}{Z_{N,\beta}^{V;[a_-,b_+]}} \leq e^{\beta N(1-\epsilon) \inf_{\xi \in [b_-, \alpha_-]} \mathcal{J}_{\min}^{V;[a_-,b_+]}(\xi)}.$$

By assumption, $\tilde{\mathcal{J}}_{\min}^{V;[b_-,b_+]}(a_-) = \mathcal{J}_{\min}^{V;[b_-,b_+]}(a_-) - \inf_{\xi \in [b_-, \alpha_-]} \mathcal{J}^{V;[b_-,b_+]}(\xi) > 0$, leading to:

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \left| \partial_{a_-} \ln Z_{N,\beta}^{V;[a_-,b_+]} \right| < 0,$$

which is the bound we sought. The arguments at the upper edge are similar. \diamond

3 Loop equations

We shall assume in this Section and also in Section 4 that the λ_i are integrated over a segment $[a_-, a_+]$ with:

Hypothesis 3.1 $-\infty < a_- < a_+ < +\infty$.

Indeed, considering finite intervals $[a_-, a_+]$ is convenient to ensure from the beginning that the Cauchy-Stieltjes transform yields functions which are holomorphic outside $[a_-, a_+]$. We also assume in this section:

Hypothesis 3.2 $V : [a_-, a_+] \rightarrow \mathbb{C}$ can be extended as a holomorphic function in some open neighborhood of $[a_-, a_+]$.

This will allow us to use complex analysis (Cauchy residue formula, moving the contours, etc.)

We shall derive the "loop equations", also called Schwinger-Dyson equations or Pastur equations [Pas72] in this context. These equations express the invariance by change of variable of an integration, up to boundary terms. We stress that these equations are exact for finite N . Although the technique is well-known, we recall the derivation here for the β matrix models with edges a_-, a_+ in order to have a self-contained presentation.

3.1 First version

Theorem 3.1 *Loop equation at rank 1. For any $x \in \mathbb{C} \setminus [a_-, a_+]$:*

$$W_2(x, x) + (W_1(x))^2 + \left(1 - \frac{2}{\beta}\right) \frac{d}{dx} \left(W_1(x)\right) + \frac{N(1 - 2/\beta) - N^2}{(x - a_-)(x - a_+)} - N \left(\oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{1}{x - \xi} \frac{(\xi - a_-)(\xi - a_+)}{(x - a_-)(x - a_+)} V'(\xi) W_1(\xi) \right) = 0.$$

$\mathcal{C}([a_-, a_+])$ is a contour surrounding $[a_-, a_+]$ in positive orientation, and included in the domain where V' is holomorphic.

Theorem 3.2 *Loop equation at rank n . Let $x_I = (x_i)_{i \in I}$ be a $(n-1)$ -uple of spectator variables in $(\mathbb{C} \setminus [a_-, a_+])^{n-1}$. For any $x \in \mathbb{C} \setminus [a_-, a_+]$:*

$$\begin{aligned} W_{n+1}(x, x, x_I) + \sum_{J \subseteq I} W_{|J|+1}(x, x_J) W_{n-|J|}(x, x_{I \setminus J}) + \left(1 - \frac{2}{\beta}\right) \frac{d}{dx} \left(W_n(x, x_I) \right) \\ - N \left(\oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{1}{x - \xi} \frac{(\xi - a_-)(\xi - a_+)}{(x - a_-)(x - a_+)} V'(\xi) W_n(\xi, x_I) \right) \\ + \frac{2}{\beta} \sum_{i \in I} \frac{d}{dx_i} \left(\frac{W_{n-1}(x, x_{I \setminus \{i\}}) - \frac{(x_i - a_-)(x_i - a_+)}{(x - a_-)(x - a_+)} W_{n-1}(x_I)}{x - x_i} \right) = 0. \end{aligned}$$

Proof of Theorem 3.1. For any smooth real-valued function h , and $\varepsilon > 0$ small enough,

$$\psi_{h,\varepsilon} : \lambda \mapsto \lambda + \varepsilon h(\lambda)$$

defines a differentiable family of diffeomorphisms from $[a_-, a_+]$ to some interval $\psi_{h,\varepsilon}([a_-, a_+])$. We assume hereafter that $h(a_-) = h(a_+) = 0$ so that $\psi_{h,\varepsilon}([a_-, a_+]) = [a_-, a_+]$ for ε small enough. We have:

$$1 = \int_{[a_-, a_+]^N} d\mu_{N,\beta}^V(\psi_{h,\varepsilon}(\lambda_1), \dots, \psi_{h,\varepsilon}(\lambda_N)). \quad (3-1)$$

When $\varepsilon \rightarrow 0$, the first subleading order of the right hand side must vanish. It can be computed in three parts. A first term comes from the variation of the Lebesgue measure $\prod_i d\lambda_i$, which is given by the Jacobian of the change of variable:

$$\left(\prod_{i=1}^N d\psi_{h,\varepsilon}(\lambda_i) \right) = \left(\prod_{i=1}^N d\lambda_i \right) \left(1 + \varepsilon \int h'(\xi) dM_N(\xi) + o(\varepsilon) \right).$$

A second term comes from the variation of the Vandermonde:

$$\begin{aligned} |\Delta(\psi_{h,\varepsilon}(\lambda))|^\beta &= |\Delta(\lambda)|^\beta \left[1 + \varepsilon \beta \sum_{1 \leq i < j \leq N} \frac{h(\lambda_i) - h(\lambda_j)}{\lambda_i - \lambda_j} + o(\varepsilon) \right] \\ &= |\Delta(\lambda)|^\beta \left\{ 1 + \varepsilon \frac{\beta}{2} \left(\iint \frac{h(\xi) - h(\eta)}{\xi - \eta} dM_N(\xi) dM_N(\eta) - \int h'(\xi) dM_N(\xi) \right) + o(\varepsilon) \right\}. \end{aligned}$$

The last term comes from the variation of the Boltzmann weight:

$$\prod_{i=1}^N e^{-\frac{N\beta}{2} V[\psi_{h,\varepsilon}(\lambda_i)]} = \left(\prod_{i=1}^N e^{-\frac{N\beta}{2} V(\lambda_i)} \right) \left(1 - \varepsilon \frac{N\beta}{2} \int V'(\xi) h(\xi) dM_N(\xi) + o(\varepsilon) \right).$$

Summing all terms up, the first order in ε in Eqn. 3-1 vanishes iff:

$$\begin{aligned} & \mu_{N,\beta}^{V;[a_-,a_+]} \left[\iint \frac{h(\xi) - h(\eta)}{\xi - \eta} dM_N(\xi) dM_N(\eta) - N \int V'(\xi) h(\xi) dM_N(\xi) \right] \\ &= \left(1 - \frac{2}{\beta}\right) \mu_{N,\beta}^{V;[a_-,a_+]} \left[\int h'(\xi) dM_N(\xi) \right]. \end{aligned} \quad (3-2)$$

Note that even though this equation was obtained for real-valued functions h , we can at this point remove this condition by linearity. To obtain an equation involving correlators, one can take for $x \in \mathbb{C} \setminus [a_-, a_+]$ the function h defined by:

$$h(\xi) = \frac{(\xi - a_-)(\xi - a_+)}{x - \xi} = \frac{(x - a_-)(x - a_+)}{x - \xi} + a_- + a_+ - x - \xi.$$

thus preserving $[a_-, a_+]$. We recall that V' is holomorphic in a neighborhood of $[a_-, a_+]$. So, by Cauchy formula, for any contour $\mathcal{C}([a_-, a_+])$ surrounding $[a_-, a_+]$ inside this neighborhood and not enclosing x :

$$\int \frac{V'(\xi)(\xi - a_-)(\xi - a_+)}{x - \xi} dM_N(\xi) = \int dM_N(\xi) \oint_{\mathcal{C}([a_-, a_+])} \frac{d\eta}{2i\pi} \frac{V'(\eta)(\eta - a_-)(\eta - a_+)}{(\eta - \xi)(x - \eta)}.$$

Hence, we obtain:

$$\begin{aligned} & W_2(x, x) + (W_1(x))^2 - \frac{N^2}{(x - a_-)(x - a_+)} \\ & - N \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{1}{x - \xi} \frac{(\xi - a_-)(\xi - a_+)}{(x - a_-)(x - a_+)} V'(\xi) W_1(\xi) \\ &= \left(1 - \frac{2}{\beta}\right) \left(-\frac{d}{dx}(W_1(x)) - \frac{N}{(x - a_-)(x - a_+)} \right). \end{aligned}$$

Proof of Theorem 3.2. By definition of the cumulants, if we define a shifted potential $V_{(x;\varepsilon)}(\xi) = V(\xi) + \frac{\varepsilon}{x - \xi}$, we have:

$$W_n^V(x, x_2, \dots, x_n) = -\frac{2}{\beta N} \partial_\varepsilon \left(W_{n-1}^{V_{(x;\varepsilon)}}(x_2, \dots, x_n) \right) \Big|_{\varepsilon=0}.$$

Notice that the matrix integral with this shifted potential is still convergent, because the eigenvalues live on the finite interval $[a_-, a_+]$. Therefore, we can obtain the loop equations at rank n by taking a perturbed potential in Thm. 3.1:

$$V_{(x_2;\varepsilon_2), \dots, (x_n;\varepsilon_n)}(\xi) = V(\xi) + \sum_{i=2}^n \frac{\varepsilon_i}{x_i - \xi},$$

and identifying the term in $\left[\prod_{i=2}^n \left(\frac{-2}{\beta N} \right) \varepsilon_i \right]$ when $\varepsilon_i \rightarrow 0$.

3.2 Second version

Here is another equivalent form of the loop equations. All W_n depend implicitly on the interval of integration $[a_-, a_+]$.

Theorem 3.3 *Loop equation at rank 1. For any $x \in \mathbb{C} \setminus [a_-, a_+]$:*

$$W_2(x, x) + (W_1(x))^2 + \left(1 - \frac{2}{\beta}\right) \frac{d}{dx} (W_1(x)) - N \left(\oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{V'(\xi) W_1(\xi)}{x - \xi} \right) - \frac{2}{\beta} \left(\frac{\partial_{a_-} \ln Z}{x - a_-} + \frac{\partial_{a_+} \ln Z}{x - a_+} \right) = 0.$$

$\mathcal{C}([a_-, a_+])$ is a contour surrounding $[a_-, a_+]$ in positive orientation, and included in the domain where V is holomorphic.

Theorem 3.4 *Loop equation at rank n . Let $x_I = (x_i)_{i \in I}$ a $(n-1)$ -uple of spectator variables in $(\mathbb{C} \setminus [a_-, a_+])^{n-1}$. For any $x \in \mathbb{C} \setminus [a_-, a_+]$:*

$$W_{n+1}(x, x, x_I) + \sum_{J \subseteq I} W_{|J|+1}(x, x_J) W_{n-|J|}(x, x_{I \setminus J}) + \left(1 - \frac{2}{\beta}\right) \frac{d}{dx} (W_n(x, x_I)) - N \left(\oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{V'(\xi) W_n(\xi, x_I)}{x - \xi} \right) + \frac{2}{\beta} \sum_{i \in I} \frac{d}{dx_i} \left(\frac{W_{n-1}(x, x_{I \setminus \{i\}}) - W_{n-1}(x_I)}{x - x_i} \right) - \frac{2}{\beta} \left(\frac{\partial_{a_-} W_{n-1}(x_I)}{x - a_-} + \frac{\partial_{a_+} W_{n-1}(x_I)}{x - a_+} \right) = 0.$$

Proof In the former proof, if we use a change of variable h which does not preserve $[a_-, a_+]$, the partition function becomes (to first order in ε):

$$Z_N^{V; \psi_{h, \varepsilon}([a_-, a_+])} \rightarrow Z_N^{V; [a_-, a_+]} \left[1 + \varepsilon \left(h(a_-) \partial_{a_-} \ln Z_N^{V; [a_-, a_+]} + h(a_+) \partial_{a_+} \ln Z_N^{V; [a_-, a_+]} \right) + o(\varepsilon) \right].$$

Thus, Eqn. 3-2 receives those extra terms, and becomes:

$$\begin{aligned} & \mu_{N, \beta}^{V; [a_-, a_+]} \left[\iint \frac{h(\xi) - h(\eta)}{\xi - \eta} dM_N(\xi) dM_N(\eta) - N \int V'(\xi) h(\xi) dM_N(\xi) \right] \\ &= \left(1 - \frac{2}{\beta}\right) \mu_{N, \beta}^{V; [a_-, a_+]} \left[\int h'(\xi) dM_N(\xi) \right] + \frac{2}{\beta} \left(h(a_-) \partial_{a_-} \ln Z_N^{V; [a_-, a_+]} + h(a_+) \partial_b \ln Z_N^{V; [a_-, a_+]} \right). \end{aligned}$$

In particular, when we choose $h(\xi) = \frac{1}{x - \xi}$, we obtain:

$$\begin{aligned} & W_2(x, x) + (W_1(x))^2 - N \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{V'(\xi) W_1(\xi)}{x - \xi} \\ &= - \left(1 - \frac{2}{\beta}\right) \frac{d}{dx} (W_1(x)) + \frac{2}{\beta} \frac{\partial_{a_-} \ln Z}{x - a_-} + \frac{2}{\beta} \frac{\partial_{a_+} \ln Z}{x - a_+}. \end{aligned}$$

The loop equation at higher rank can be deduced as before by perturbing the potential.

◇

3.3 Remark

If we compare those expressions to the first version of the loop equations, we find by consistency:

$$\partial_{a_\tau} \ln Z = \frac{1}{a_{-\tau} - a_\tau} \left\{ -\frac{N^2\beta}{2} + N \left(\frac{\beta}{2} - 1 \right) + \frac{N\beta}{2} \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} (\xi - a_{-\tau}) V'(\xi) W_1(\xi) \right\},$$

and for higher correlators $\partial_{a_\tau} W_{n-1}(x_I)$ equals

$$\frac{1}{a_{-\tau} - a_\tau} \left\{ \frac{N\beta}{2} \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} (\xi - a_{-\tau}) V'(\xi) W_n(\xi) + \sum_{i \in I} \frac{d}{dx_i} \left((x_i - a_{-\tau}) W_{n-1}(x_I) \right) \right\}$$

for $\tau \in \{\pm\}$.

4 The $1/N$ expansion

4.1 Notations, assumptions, proposition

This section relies on complex analysis and inequalities for probability measures. We make four assumptions on the potential V , which are valid only in this section. The link with our main theorem will be done in Section 5.

We keep on with the assumption:

Hypothesis 4.1 $-\infty < a_- < a_+ < +\infty$.

Since V is smooth, the equilibrium measure $d\mu_{\text{eq}}^{V;[a_-, a_+]}$ (ξ) will in fact be a density $\rho(\xi)d\xi$, where $\rho : [a_-, a_+] \rightarrow [0, +\infty]$ is a continuous function. We call $\text{supp } \rho = \overline{\{x \in [a_-, a_+] \mid \rho(x) > 0\}}$ its support. In the hermitian case ($\beta = 2$), a $1/N$ expansion is expected only when $\text{supp } \rho$ is connected. We assume here also:

Hypothesis 4.2 V leads to a one-cut regime, i.e. the support of $\mu_{\text{eq}}^{V;[a_-, a_+]}$ is an interval $[\alpha_-, \alpha_+] \subseteq [a_-, a_+]$.

In order to write the loop equations as in Section 3, we assume:

Hypothesis 4.3 V is real-valued on $[a_-, a_+]$, and can be extended as a holomorphic function on some open neighborhood U of $[a_-, a_+]$.

We justify in Remark 4.1 later that there exists a unique analytic function $y : U \rightarrow \mathbb{C} \cup \{\infty\}$ such that, for any $x \in [\alpha_-, \alpha_+]$, we have:

$$\rho(x) = \frac{1}{i\pi} \lim_{\epsilon \rightarrow 0^+} y(x + i\epsilon). \quad (4-1)$$

This function can be written $y(x) = S(x)\sigma(x)$, where S is now a holomorphic function defined on U , and σ is of the form:

$$\sigma(x) = \sqrt{\frac{\prod_{\tau \in \text{Soft}}(x - \alpha_\tau)}{\prod_{\tau' \in \text{Hard}}(x - \alpha_{\tau'})}}. \quad (4-2)$$

The lower edge a_- is

- either a hard edge, meaning that $a_- = \alpha_-$. Then, $\rho(x) \in O((x - \alpha_-)^{-1/2})$ when $x \rightarrow \alpha_-$.
- or a soft edge, meaning that $a_- < \alpha_-$. Then, $\rho(x) \in O((x - \alpha_-)^{1/2})$ when $x \rightarrow \alpha_-$.

and the same distinction exists independently for the upper edge a_+ . Our discussion holds for both hard and soft cases. However, a key technical assumption is:

Hypothesis 4.4 *V is offcritical on $[a_-, a_+]$, in the sense that $S(x)$ remains positive on $[a_-, a_+]$.*

For instance, Hyp. 4.2 and 4.4 automatically hold when V is strictly convex. For a generic V satisfying Hyp. 4.2, we have $S(\alpha_-) > 0$ and $S(\alpha_+) > 0$, so we can always find an interval $[a_-, a_+]$ which is a strict enlargement of $[\alpha_-, \alpha_+]$, such that Hyp. 4.4 holds on $[a_-, a_+]$. We call "critical point on $[a_-, a_+]$ ", the situation corresponding to a choice of V such that S has a zero on $[a_-, a_+]$. In this article, we do not tackle the question of the double scaling limit for β matrix models ($N \rightarrow +\infty$ and coefficients of V finely tuned with N to achieve a critical point when $N = \infty$). Though, this would be a very interesting regime in relation with universality questions, considering the absence of Riemann-Hilbert techniques when $\beta \neq 1, 2, 4$.

We shall allow V itself to depend on N and have a $1/N$ expansion. To give precise statements about those expansions, we need some notations. For any Jordan curve Γ , we note $\text{Ext}(\Gamma)$ (resp. $\text{Int}(\Gamma)$) the unbounded (resp. bounded) connected component of $\mathbb{C} \setminus \Gamma$. In the following, we fix once for all a Jordan curve Γ_E , and a sequence of nested Jordan curves $(\Gamma_l)_{l \in \mathbb{N}}$, which all live in $\mathbb{C} \setminus [a_-, a_+]$, and such that

- (i) $\Gamma_E \subseteq U$.
- (ii) $\{x \in U \mid S(x) = 0\} \cap \text{Int}(\Gamma_E) = \emptyset$.
- (iii) $\forall l \in \mathbb{N} \quad \Gamma_l \subseteq \text{Int}(\Gamma_{l+1})$.
- (iv) $\forall l \in \mathbb{N} \quad \Gamma_l \subseteq \text{Int}(\Gamma_E)$.

The contour configuration is depicted in Fig. 1, where the zeroes of S were called s_i . In the remaining of the text, Γ will refer to a Jordan curve in $\text{Int}(\Gamma_E) \setminus [a_-, a_+]$. We will use the following norm on the space of continuous functions on a contour Γ

$$\|f\|_\Gamma = \sup_{x_i \in \Gamma} |f(x_1, \dots, x_n)|.$$

On the space $\mathcal{H}_{n;[a_-, a_+]}^{(1)}$ of holomorphic functions on $(\mathbb{C} \setminus [a_-, a_+])^n$, which behave as $O(1/x_i)$ when $x_i \rightarrow \infty$, we have by the maximum principle

$$\|f\|_\Gamma = \sup_{x_i \in \text{Ext}(\Gamma)} |f(x_1, \dots, x_n)|.$$

One can easily derive the following useful inequalities:

$$\forall f \in \mathcal{H}_{1;[a_-, a_+]}^{(1)} \quad \forall x_0 \in \text{Ext}(\Gamma_{l+1}) \quad \forall l \in \mathbb{N} \quad \left\| \frac{f(\bullet) - f(x_0)}{\bullet - x_0} \right\|_{\Gamma_l} \leq \|f'\|_{\Gamma_{l+1}} \leq \zeta_l \|f\|_{\Gamma_l}, \quad (4-3)$$

where $\zeta_l = \frac{\ell(\Gamma_l)}{2\pi d^2(\Gamma_l, \Gamma_{l+1})}$ is a finite constant depending only on the relative position of Γ_l and Γ_{l+1} .

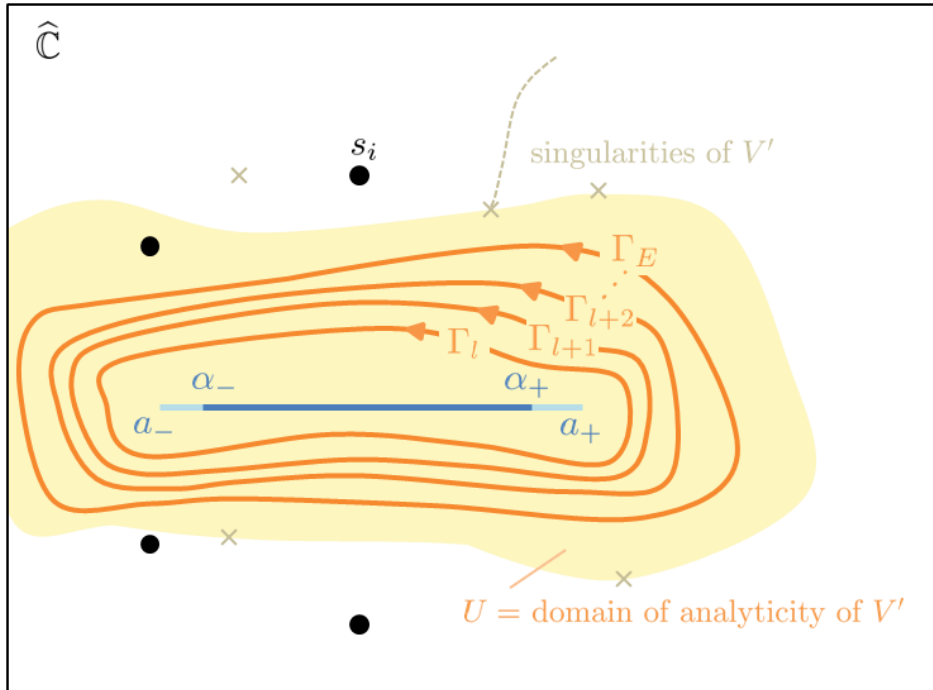


Figure 1: Hypothesis on the location of s_i and contour configurations

Now, we can state our last assumption:

Hypothesis 4.5 V admits a $1/N$ asymptotic expansion:

$$V(x) \underset{\Gamma_E}{=} \sum_{k \geq 0} N^{-k} V^{\{k\}}(x),$$

with functions $V^{\{k\}}$ independent of N , such that $V^{\{k\}}$ is holomorphic in U . The $\underset{\Gamma_E}{=}$ equality means that, for any positive integer K , there exists a positive constant v_K such that, for N large enough:

$$\sup_{\xi \in \Gamma_E} \left| V(\xi) - \sum_{k=0}^K N^{-k} V^{\{k\}}(\xi) \right| \leq N^{-(K+1)} v_K.$$

(The maximum principle implies automatically the same bound with Γ replacing Γ_E as V is analytic in $\text{Int}(\Gamma_E)$).

In many applications, V is independent of N (i.e. $V \equiv V^{\{0\}}$). There is however no difficulty in our reasoning to consider potentials which depend on N within Hyp. 4.5.

Our intermediate result is:

Theorem 4.1 If Hyp. 4.1-4.5 hold, the correlators admit an asymptotic expansion when $N \rightarrow \infty$ with respect to the norm $\|\cdot\|_{\Gamma_E}$, of the form:

$$\forall n \geq 1, \quad W_n = \sum_{k \geq n-2} N^{-k} W_n^{\{k\}},$$

where $W_n^{\{k\}} \in \mathcal{H}_{n;[\alpha_-, \alpha_+]}^{(1)}$.

4.2 Relevant linear operators

4.2.1 The operator \mathcal{K}

We introduce the following linear operator defined on the space $\mathcal{H}_{1;[a_-, a_+]}^{(2)}$ of holomorphic functions on $\mathbb{C} \setminus [a_-, a_+]$ which behave as $O(1/x^2)$ when $x \rightarrow \infty$:

$$(\mathcal{K}f)(x) = 2W_1^{\{-1\}}(x)f(x) - \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{L(\xi)}{L(x)} \left(\frac{1}{x-\xi} + c \right) (V^{\{0\}})'(\xi) f(\xi).$$

This operator for an appropriate choice of L and c appears in the loop equations. We have found the following choice convenient:

$$L(x) = \prod_{\tau \in \text{Hard}} (x - a_\tau).$$

$$c = \begin{cases} 0 & \text{if Soft} = \{\pm\} \text{ or Hard} = \{\pm\}, \\ \frac{1}{a_\tau - a_{-\tau}} & \text{if } \tau \in \text{Soft} \text{ and } (-\tau) \in \text{Hard}. \end{cases}$$

We may also rewrite:

$$(\mathcal{K}f)(x) = -2y(x)f(x) + \frac{(\mathcal{Q}f)(x)}{L(x)}, \quad (4-4)$$

with:

$$(\mathcal{Q}f)(x) = - \oint_{\mathcal{C}([a_-, a_+]) \cup \mathcal{C}(x)} \frac{d\xi}{2i\pi} L(\xi) \left(\frac{1}{x-\xi} + c \right) (V^{\{0\}})'(\xi) f(\xi),$$

where $\mathcal{C}(x)$ is a contour surrounding x only (computing a residue at x). For any $f \in \mathcal{H}_{1;[a_-, a_+]}^{(1)}$, $(\mathcal{Q}f)$ is analytic, with singularities only where $(V^{\{0\}})'$ has singularities, in particular is holomorphic in the neighborhood of $[a_-, a_+]$. We have set:

$$y(x) = -W_1^{\{-1\}}(x) + \frac{(V^{\{0\}})'(x)}{2}.$$

y is discontinuous on the support of $\mu_{\text{eq}}^{V;[a_-, a_+]}$ (see Thm. 1.1), i.e. on $[\alpha_-, \alpha_+] \subseteq [a_-, a_+]$, but analytic on $\mathbb{C} \setminus [\alpha_-, \alpha_+]$. We justify in Remark 4.1 that $y(x) = S(x)\sigma(x)$ where $\sigma(x)$ was introduced in Eqn. 4-2 and the squareroot is chosen with its usual discontinuity on \mathbb{R}_- . Let us call $s_i \neq \alpha_-, \alpha_+$ the zeroes of $S(x)$ in the complex plane, and we assume that they do not lie in $[a_-, a_+]$ (Hyp. 4.4).

It is clear that $\text{Im } \mathcal{K} \subseteq \mathcal{H}_{1;[a_-, a_+]}^{(1)}$. Here, $W_1^{\{-1\}}$ (hence y) has only cut $[\alpha_-, \alpha_+]$, and this operator is invertible¹. Its inverse can be explicitly written, it is given by Tricomi formula [Tri57]:

$$\forall x \in \mathbb{C} \setminus [a_-, a_+], \forall g \in \text{Im } \mathcal{K}, \quad (\mathcal{K}^{-1}g)(x) = \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{1}{\xi-x} \frac{\tilde{\sigma}(\xi)}{\tilde{\sigma}(x)} \frac{g(\xi)}{2y(\xi)}. \quad (4-5)$$

where $\tilde{\sigma}(x) = \sqrt{(x-\alpha_-)(x-\alpha_+)}$, and where we integrate over a contour surrounding $[a_-, a_+]$ but not x . Indeed, if $g \in \text{Im } \mathcal{K}$, we can write for any $x \in \mathbb{C} \setminus [a_-, a_+]$:

$$\begin{aligned} \tilde{\sigma}(x)f(x) &= \text{Res}_{\xi \rightarrow x} \frac{d\xi}{\xi-x} \tilde{\sigma}(\xi) f(\xi) \\ &= - \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{\tilde{\sigma}(\xi) f(\xi)}{\xi-x} \\ &= - \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{\tilde{\sigma}(\xi)}{\xi-x} \frac{1}{2y(\xi)} \left(-g(\xi) + \frac{(\mathcal{Q}f)(\xi)}{L(\xi)} \right) \\ &= \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{\tilde{\sigma}(\xi)}{\xi-x} \frac{g(\xi)}{2y(\xi)}. \end{aligned}$$

In the second line, we moved the contour from a neighborhood of x to a neighborhood of $[a_-, a_+]$, and used the fact that $\tilde{\sigma}(\xi) \in O(\xi)$ and $f(\xi) \in O(1/\xi^2)$, so that the residue

¹In general, on the space of holomorphic functions with $g+1$ cuts, $\dim \text{Ker } \mathcal{K} = g$, and one has to prescribe g cycle integrals in order to define an inverse operator.

at ∞ of the integrand vanishes. In the fourth line, we use the fact that L is chosen such that $\frac{\tilde{\sigma}(\xi)}{y(\xi)L(\xi)} = \frac{1}{S(\xi)}$, which is holomorphic in a neighborhood of $[a_-, a_+]$ thanks to Hyp. 4.4. Since $(\mathcal{Q}f)$ is also holomorphic in a neighborhood of $[a_-, a_+]$, the contour integral of this term vanishes. For our purposes, it is not necessary to describe the vector space $\text{Im } \mathcal{K}$. Notice that if we apply \mathcal{K}^{-1} to a function $g \in \text{Im } \mathcal{K}$ which is furthermore holomorphic outside $\mathbb{C} \setminus [a_-, a_+]$, we can contract the contour $\mathcal{C}([a_-, a_+])$ to a contour $\mathcal{C}([\alpha_-, \alpha_+])$.

4.2.2 Continuity of \mathcal{K} and \mathcal{K}^{-1}

The key fact in this article is that \mathcal{K}^{-1} is a continuous operator in $(\text{Im } \mathcal{K}, \|\cdot\|_\Gamma)$:

Lemma 4.1 *Im \mathcal{K} is closed subspace of $\mathcal{H}_{1;[a_-, a_+]}^{(1)}$ for the topology induced by the norm $\|\cdot\|_\Gamma$, and there exists a constant $k > 0$, such that:*

$$\forall g \in \text{Im } \mathcal{K}, \quad \|\mathcal{K}^{-1} g\|_\Gamma \leq k \|g\|_\Gamma.$$

We call $\|\mathcal{K}^{-1}\|_\Gamma$, the infimum of such constants k .

Proof. Let us prove first that \mathcal{K} , as a endomorphism of $\mathcal{H}_{1;[a_-, a_+]}^{(1)}$, is continuous. For any $f \in \mathcal{H}_{1;[a_-, a_+]}^{(1)}$ in formula 4-4, if x runs along Γ , we can move the contour $\mathcal{C}([a_-, a_+]) \cup \mathcal{C}(x)$ to Γ_E and get the bound:

$$\begin{aligned} \|\mathcal{K}f\|_\Gamma &\leq 2(\|y\|_\Gamma \|f\|_\Gamma + \frac{\ell(\Gamma_E)}{2\pi} \frac{\|L\|_{\Gamma_E}}{\min_{x \in \Gamma} |L(x)|} \left(\frac{1}{d(\Gamma_E, \Gamma)} + c \right) \|(V^{\{0\}})'\|_{\Gamma_E} \|f\|_{\Gamma_E} \\ &\leq \left[2\|y\|_\Gamma + \frac{\ell(\Gamma_E)}{2\pi} \frac{\|L\|_{\Gamma_E}}{\min_{x \in \Gamma} |L(x)|} \left(\frac{1}{d(\Gamma, \Gamma_E)} + c \right) \|(V^{\{0\}})'\|_{\Gamma_E} \right] \|f\|_\Gamma. \end{aligned} \quad (4-6)$$

We have used again the maximum principle for f to find the second line. Likewise, we can show that $\mathcal{K}^{-1} : \text{Im } \mathcal{K} \rightarrow \mathcal{H}_{1;[a_-, a_+]}^{(1)}$ is continuous. In formula 4-5, we put x on Γ , and move the contour from $\mathcal{C}([a_-, a_+])$ to Γ_E in Eqn. 4-5. Doing so, we pick up a simple pole at $\xi = x$, and we find:

$$(\mathcal{K}^{-1}g)(x) = -\frac{g(x)}{2y(x)} + \frac{1}{\tilde{\sigma}(x)} \oint_{\Gamma_E} \frac{d\xi}{2i\pi} \frac{1}{\xi - x} \frac{L(\xi)g(\xi)}{2S(\xi)}.$$

We deduce :

$$\begin{aligned} \|\mathcal{K}^{-1}g\|_\Gamma &\leq \frac{\|g\|_\Gamma}{2 \min_{x \in \Gamma} |y(x)|} + \frac{\ell(\Gamma_E)}{4\pi d(\Gamma, \Gamma_E)} \frac{\max_{\xi \in \Gamma_E} |L(\xi)|}{\min_{x \in \Gamma} |\tilde{\sigma}(x)|} \frac{\|g\|_{\Gamma_E}}{\min_{\xi \in \Gamma_E} |S(\xi)|} \\ &\leq \left(\frac{1}{2 \min_{x \in \Gamma} |y(x)|} + \frac{\ell(\Gamma_E)}{4\pi d(\Gamma, \Gamma_E)} \frac{\|L\|_{\Gamma_E}}{\min_{\xi \in \Gamma} |\tilde{\sigma}(x)| \min_{\xi \in \Gamma} |S(\xi)|} \right) \|g\|_\Gamma. \end{aligned} \quad (4-7)$$

where we used the maximum principle in the last line. Eventually, let us show that $\text{Im } \mathcal{K}$ is a closed subspace of $\mathcal{H}_{1;[a_-, a_+]}^{(1)}$. We pick up a sequence $(g_n)_n$ in $\text{Im } \mathcal{K}$ converging

towards $g \in \mathcal{H}_{1;[a_-,a_+]}^{(1)}$ for a norm $\|\cdot\|_{\Gamma_0}$ on a given contour Γ_0 . Let $(f_n)_n$ be a sequence in $\mathcal{H}_{1;[a_-,a_+]}^{(1)}$ such that $g_n = \mathcal{K}f_n$, or equivalently $f_n = \mathcal{K}^{-1}g_n$. Using Eqn. 4-7 for any contour Γ , we know that $\|f_n\|_{\Gamma} \leq k \|g_n\|_{\Gamma}$ for some constant $k > 0$. So, f_n is a locally bounded subsequence of holomorphic functions in $\mathbb{C} \setminus [a_-, a_+]$. By Montel's theorem, it admits a subsequence $(f_{\phi(n)})_n$ converging to some $f \in \mathcal{H}_{1;[a_-,a_+]}^{(1)}$ uniformly on any compact of $\mathbb{C} \setminus [a_-, a_+]$. Then using Eqn. 4-6, $g_{\phi(n)} = \mathcal{K}f_{\phi(n)} \rightarrow \mathcal{K}f$ for the norm $\|\cdot\|_{\Gamma_0}$. In particular, $g(x) = \mathcal{K}f(x)$ for all $x \in \text{Ext}(\Gamma_0)$. Since g and f are both analytic in $\mathbb{C} \setminus [a_-, a_+]$, they must coincide on $\mathbb{C} \setminus [a_-, a_+]$. Hence, $g \in \text{Im } \mathcal{K}$, showing that $\text{Im } \mathcal{K}$ is closed. \diamond

$\|\mathcal{K}^{-1}\|_{\Gamma}$ is controlled by the distance of the zeroes s_i to the support $[a_-, a_+]$. This motivates Hyp. 4.4.

4.2.3 The endomorphism "negative part"

Let g be a holomorphic function at least in a neighborhood of $[a_-, a_+]$. The following endomorphism of $\mathcal{H}_{1;[a_-,a_+]}^{(1)}$ often appears in the loop equations:

$$\mathcal{N}_g(f)(x) = \oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi} \frac{L(\xi)}{L(x)} \left(\frac{1}{x-\xi} + c \right) g(\xi) f(\xi).$$

We may write sometimes $\mathcal{N}_g[f(x)]$ as an abuse of notation. The analyticity assumption on g ensures that \mathcal{N}_g is a continuous operator with respect to the norm $\|\cdot\|_{\Gamma}$. Indeed, let us put x on Γ and move the contour $\mathcal{C}([a_-, a_+])$ to Γ_E :

$$\mathcal{N}_g(f)(x) = g(x)f(x) + \oint_{\Gamma_E} \frac{d\xi}{2i\pi} \frac{L(\xi)}{L(x)} \left(\frac{1}{x-\xi} + c \right) g(\xi) f(\xi).$$

Thus, the maximum principle implies:

$$\begin{aligned} \|\mathcal{N}_g(f)\|_{\Gamma} &\leq \|g\|_{\Gamma} \|f\|_{\Gamma} + \frac{\ell(\Gamma_E)}{2\pi} \left(\frac{1}{d(\Gamma_E, \Gamma)} + |c| \right) \frac{\max_{\xi \in \Gamma_E} |L(\xi)|}{\min_{x \in \Gamma} |L(x)|} \|g\|_{\Gamma_E} \|f\|_{\Gamma_E} \\ &\leq \left[\|g\|_{\Gamma} + \frac{\ell(\Gamma_E)}{2\pi} \left(\frac{1}{d(\Gamma_E, \Gamma)} + |c| \right) \frac{\max_{\xi \in \Gamma_E} |L(\xi)|}{\min_{x \in \Gamma} |L(x)|} \|g\|_{\Gamma_E} \right] \|f\|_{\Gamma}. \end{aligned}$$

4.3 Order of magnitude of W_n

If there exists a $1/N$ expansion, W_n ought to be of order of magnitude N^{2-n} . Let us start with a lemma explaining how this can be inferred from rough bounds on W_n . Hereafter, $O_l(\dots)$ or $o_l(\dots)$ mean $O(\dots)$ or $o(\dots)$ with respect to the norm $\|\cdot\|_{\Gamma_l}$. Since the contours Γ_l are ordered from the interior to the exterior, being a $o_{l+1}(\dots)$ is weaker than being a $o_l(\dots)$. When the index l is not precised, it is understood that the bound holds for any integer l .

Lemma 4.2 *Let $\delta_{-1}W_1 := N^{-1}W_1 - W_1^{\{-1\}}$ and $l \geq 0$. Assume $\delta_{-1}W_1 \in o_l(1)$, and for all integer $n \geq 2$, assume $W_n \in O_l(N)$. Then:*

$$\forall n \geq 2 \quad \|W_n\|_{\Gamma_{4n-6+l}} \in O(N^{2-n}).$$

Proof. Let $\delta_0V = V - V^{\{0\}}$. Firstly, as $\delta_{-1}W_1$ and $(\delta_0V)'$ goes to 0 uniformly on Γ_{-1} when $N \rightarrow \infty$, we observe that for any fixed integer k , and N large enough:

$$\begin{aligned} (1 - \varepsilon_{N,k+1})\|W_n\|_{\Gamma_{k+1}} &\leq \|K^{-1}\|_{\Gamma_{k+1}} \left\| \left[K + \delta K + \frac{1}{N} \left(1 - \frac{2}{\beta} \right) \frac{d}{dx} \right] W_n \right\|_{\Gamma_{k+1}} \\ &\quad + \frac{1}{N} \left| 1 - \frac{2}{\beta} \right| \zeta_k \|W_n\|_{\Gamma_k}, \end{aligned} \quad (4-8)$$

where

$$\begin{aligned} [\delta K](f)(x) &= -\mathcal{N}_{(\delta_0V)'}[f(x)] + 2(\delta_{-1}W_1)(x) f(x), \\ \varepsilon_{N,k+1} &= \|K^{-1}\|_{\Gamma_{k+1}} (\|\mathcal{N}_{(\delta_0V)'}\|_{\Gamma_{k+1}} + 2\|\delta_{-1}W_1\|_{\Gamma_{k+1}}) \end{aligned} \quad (4-9)$$

goes to zero as N goes to infinity for $k+1 \geq l$ by assumption. ζ_k is defined in Eqn. 4-3. We assume hereafter that N is large enough so that $\varepsilon_{N,k+1}$ is smaller than $1/2$.

Secondly, the first version of the loop equation at rank $n \geq 2$ (Thm. 3.2) can be rewritten:

$$\left[K + \delta K + \frac{1}{N} \left(1 - \frac{2}{\beta} \right) \frac{d}{dx} \right] W_n(x, x_I) = A_{n+1} + B_n + C_{n-1} + D_{n-1},$$

where:

$$\begin{aligned} A_{n+1} &= -\frac{1}{N} W_{n+1}(x, x, x_I), \\ B_n &= -\frac{1}{N} \sum_{\substack{n_1, n_2 \geq 1 \\ n_1 + n_2 = n-1}} \sum_{\substack{J \subseteq I \\ |J|=n_1}} W_{n_1+1}(x, x_J) W_{n_2+1}(x, x_{I \setminus J}), \\ C_{n-1} &= -\frac{1}{N} \frac{2}{\beta} \sum_{i \in I} \frac{d}{dx_i} \left\{ \frac{W_{n-1}(x, x_{I \setminus \{i\}})}{x - x_i} - \frac{L(x_i)}{L(x)} \left(\frac{1}{x - x_i} + c \right) W_{n-1}(x_I) \right\}, \\ D_{n-1} &= \frac{1}{N} \frac{2}{\beta} \sum_{\tau \in \text{Soft}} \frac{\partial_{a_\tau} W_{n-1}(x_I)}{x - a_\tau}. \end{aligned}$$

We know from Proposition 2.3 that $D_{n-1} \in O(e^{-Nn})$, so this term does not contribute at any order of magnitude N^{-k} . Now, if we assume that $W_n \in O_l(N)$ for all $n \geq 2$ (this is obviously true for $n = 1$), we always have $A_{n+1} \in O_l(1)$ and $C_{n-1} \in O_{l+2}(1)$, whereas the last last term in Eqn. 4-8 is bounded by hypothesis for $k \geq l$.

Now, we want to bound W_n by induction on n . At rank $n = 2$, we have $B_2 = 0$, and we deduce from Eqn. 4-8 that $W_2 \in O_{l+2}(1)$. Then at rank $n = 3$, the product

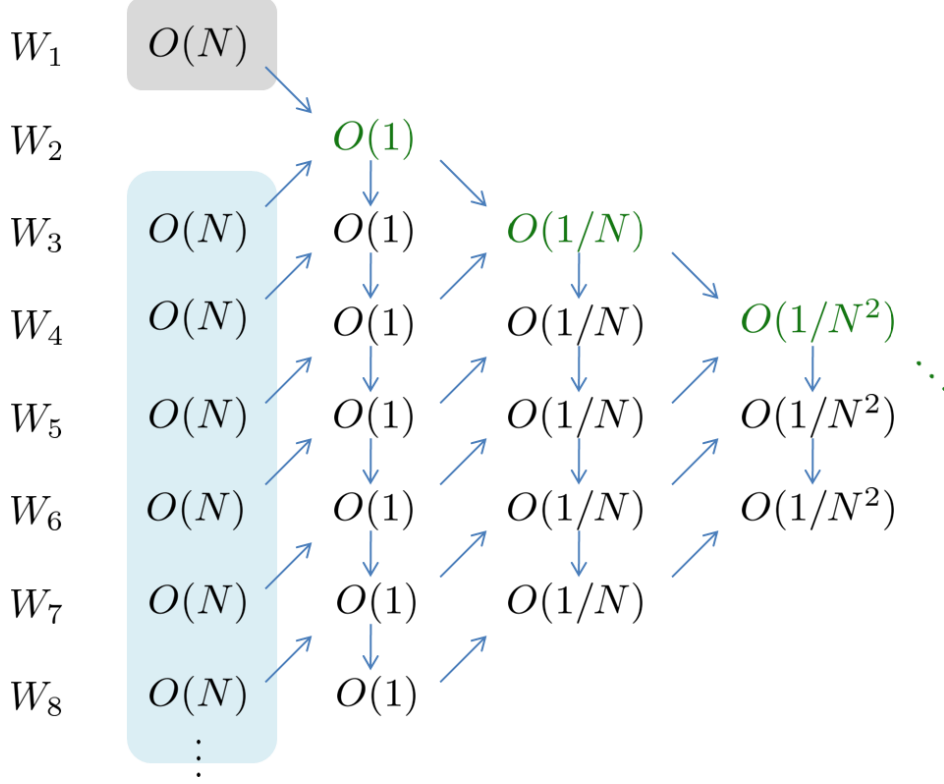


Figure 2: Triangular recursion for Lemma 4.2

term B_3 is $O_{l+2}(1/N)$ and C_2 is $O_{l+4}(1/N)$, thus $W_3 \in O_{l+4}(A_4) = O_6(1)$. Then similarly at rank $n = 4$, the product term B_4 is $O_{l+4}(1/N)$ and C_3 is $O_{l+6}(1/N)$, thus $W_4 \in O_{l+6}(1)$. This implies in return that $A_4 \in O_{l+6}(1/N)$, thus $W_3 \in O_{l+6}(1/N)$. And so on \dots . The result can be proved by a triangular induction, as depicted in Fig. 2. At each vertical step, we are forced to trade the contour Γ_k with the exterior contour Γ_{k+2} in order to control the C terms. So, to go from $W_n \in O_{k_n}(N^{2-n})$ (in the n^{th} column) to $W_n \in O_{k_{n+1}}(N^{2-(n+1)})$ (in the $(n+1)^{\text{th}}$ column), we must reach W_{n+2} in the n^{th} column. This is done by two vertical steps, thus $k_{n+1} = k_n + 4$. Since $k_2 = l + 2$, we have $k_n = 4n - 6 + l$ for all $n \geq 2$. \diamond

Lemma 4.3 *If there exists $\gamma \in [0, 1[$ and $\delta \in [0, \infty[$ such that $W_n \in O_0(N^{\gamma n - \delta})$ for all $n \geq 2$, then:*

$$\|W_n\|_{\Gamma_{4n-6+l}} \in O(N^{2-n}),$$

where $l = 2\lceil(\gamma^{-1} - 1)^{-1}\rceil$.

Proof. Now, let us rather assume the existence of $\gamma \in [0, 1[$ and $\delta \geq 0$ such that, for all $n \geq 2$, $W_n \in O_0(N^{\gamma n - \delta})$. D_{n-1} being always exponentially small, it does

not matter in our discussion. At rank $n = 2$, as $B_2 = 0$ and $C_1 \in O_2(1)$, we have $W_2 \in O_2(\max[\frac{1}{N} W_3, 1])$. We also have for all n :

$$\begin{aligned} A_{n+1} &\in O_0(\max[N^{\gamma n - \delta - (1-\gamma)}, 1]), \\ B_n &\in O_0(\max[N^{\gamma n - 2\delta - (1-\gamma)}, 1]), \\ C_{n-1} &\in O_2(\max[N^{\gamma n - \delta - (1+\gamma)}, 1]). \end{aligned}$$

When these $O(\dots)$ decay, it does not hurt to consider them as $O(1)$. So, our bounds are upgraded at least to $W_n \in O_2(\max[N^{\gamma n - \delta'}, 1])$ with $\delta' = \delta + 1 - \gamma > \delta$. By repeating the argument k times, we obtain for all $n \geq 2$, $W_n \in O_{2k}(\max[N^{((k+1)\gamma - k)n - \delta}, 1])$. Since $\gamma < 1$, by choosing an integer $k \geq \frac{1}{1-\gamma}$, we deduce that $W_n \in O_l(1)$ for all $n \geq 2$ with $l = 2k$, and we apply Lemma 4.2 to conclude. \diamond

4.4 Initialization

We now establish a priori control on the correlators. We shall use:

Lemma 4.4 *Let $w_N = N^\epsilon$ for some $\epsilon > 0$. Assume that for any integer p , there exists $C_p > 0$ and independent of N , such that for all $x \in \mathbb{C} \setminus [a_-, a_+]$:*

$$\mu_{N,\beta}^V \left\{ \left| \int \frac{dM_N(\xi)}{x - \xi} - \mu_{N,\beta}^V \left[\int \frac{dM_N(\xi)}{x - \xi} \right] \right|^p \right\} \leq \frac{C_p w_N^p}{(d(x, [a_-, a_+]))^{2p}}. \quad (4-10)$$

Then, for all $n \geq 2$, $W_n \in O(w_N^n)$ for the norm $\|\cdot\|_\Gamma$, when $N \rightarrow \infty$.

Proof. For $n \geq 2$, $W_n(x_1, \dots, x_n)$ is a polynomial in:

$$\mu_{N,\beta}^{V;[a_-, a_+]} \left\{ \prod_{j \in J} \left(\int \frac{dM_N(\xi)}{x_j - \xi} - \mu_{N,\beta}^V \left[\int \frac{dM_N(\xi)}{x_j - \xi} \right] \right) \right\},$$

with $J \subseteq \{1, \dots, n\}$, and the coefficients of this polynomial are independent of N . Thus by Eqn. 4-10 and Hölder inequality, there exists $D_n \in \mathbb{R}_+^*$ independent of N such that:

$$|W_n(x_1, \dots, x_n)| \leq \frac{D_n w_N^n}{(\min_{1 \leq i \leq n} d(x_i, [a_-, a_+]))^{2n}}.$$

Hence, taking the sup for $x_i \in \Gamma$, we find $W_n \in O(w_N^n)$.

Lemma 4.5 *Under the five assumptions of Section 4.1, Eqn. 4-10 holds for any $\epsilon > 0$.*

Proof. Our starting point comes from a result of Boutet de Monvel, Pastur and Shcherbina [dMPS95], developed by Johansson²[Joh98, (3.49)] and more recently in

²Johansson's has written his proof in the framework $[a_-, a_+] = \mathbb{R}$, but there is no difficulty adapting it to $[a_-, a_+]$ finite. Im z should be replaced by $d(z, [a_-, a_+])$, and its powers in the bound of his Lemma 3.10 and 3.11 may differ, but the order of magnitude ω_N (our w_N) is the same.

[KS10, (2.26)]. Let $\Gamma' \subseteq \text{Int } \Gamma$ be a contour surrounding $[a_-, a_+]$. For any $\phi : \text{Int}(\Gamma) \rightarrow \mathbb{C}$ which is a continuous function, and real-valued on $[a_-, a_+]$, there exists a positive constant C such that:

$$\mu_{N,\beta}^V \left[\exp \left(\frac{1}{2(\sup_{z' \in \Gamma'} |\phi(z')|) w_N} \left(\int \phi(\xi) dM_N(\xi) - N \int \phi(\xi) dL(\xi) \right) \right) \right] \leq 3,$$

where $w_N = C \ln N$. By Chebychev's inequality, we deduce that:

$$\forall t \in [0, +\infty[, \quad \mu_{N,\beta}^V \left\{ \left| \int \phi(\xi) dM_N(\xi) - N \int \phi(\xi) dL(\xi) \right| \geq t (\sup_{z' \in \Gamma'} |\phi(z')|) w_N \right\} \leq 6e^{-t},$$

and therefore for all $p \in \mathbb{N}$,

$$\mu_{N,\beta}^V \left[\left| \int \phi(\xi) dM_N(\xi) - N \int \phi(\xi) dL(\xi) \right|^p \right] \leq p! (\sup_{z' \in \Gamma'} |\phi(z')|)^p w_N^p.$$

In particular, we can apply this discussion to $\phi(z) = \text{Re} \frac{1}{x-z}$ and $\phi(z) = \text{Im} \frac{1}{x-z}$ where x is a point of Γ . This leads to Eqn. 4-10. \diamond

In the case of a strictly convex potential, we may use instead concentration of measure:

Lemma 4.6 *If $V^{\{0\}}$ is strictly convex on $[a_-, a_+]$, then Eqn. 4-10 holds with $\epsilon = 0$.*

Proof. Since $V^{\{0\}}$ is strictly convex on $[a_-, a_+]$, V is also strictly convex on $[a_-, a_+]$ for N large enough. By concentration of measure, see [GZ00] or [AGZ10, Section 2.3 and Exercise 4.4.33], there exists $c > 0$ such that, for all $x \in \mathbb{C} \setminus [a_-, a_+]$, for all $\epsilon > 0$ and $N \in \mathbb{N}$:

$$\mu_{N,\beta}^V \left\{ \left| \int \frac{dM_N(\xi)}{x - \xi} - \mu_{N,\beta}^V \left[\int \frac{dM_N(\xi)}{x - \xi} \right] \right| \geq \frac{\epsilon}{(d(x, [a_-, a_+]))^2} \right\} \leq 2e^{-c\epsilon^2}.$$

This entails Eqn. 4-10. \diamond

4.5 Leading order of W_1

Afterwards, all steps only rely on the analysis of loop equations. Although we already know the characterization of the equilibrium measure μ_{eq} , and thus of its Stieltjes transform $W_1^{\{-1\}}$, let us recall how $W_1^{\{-1\}}$ is characterized by the loop equations. We write the loop equation at rank 1 (Thm. 3.1):

$$\begin{aligned}
& \frac{1}{N^2} W_2(x, x) \\
& + (W_1^{\{-1\}}(x))^2 - \frac{1}{(x-a_-)(x-a_+)} \\
& - \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{1}{x-\xi} \frac{(\xi-a_-)(\xi-a_+)}{(x-a_-)(x-a_+)} (V^{\{0\}})'(\xi) W_1^{\{-1\}}(\xi) \\
+ \mathcal{K}[\delta_{-1}W_1](x) & + \frac{1}{N} \left(1 - \frac{2}{\beta}\right) \left(W_1^{\{-1\}}(x) + \frac{1}{(x-a_-)(x-a_+)}\right) - \frac{1}{N} \mathcal{N}_{V^{\{1\}}} [W_1^{\{-1\}}](x) \\
& + ((\delta_{-1}W_1)(x))^2 - \mathcal{N}_{(\delta_0V)'} [\delta_{-1}W_1](x) + \frac{1}{N} \mathcal{N}_{(\delta_1V)'} [W_1^{\{-1\}}](x) = 0.
\end{aligned} \tag{4-11}$$

We already know that the 4th and the 5th line are $o(1)$. Since $W_2 \in o(N^2)$, $W_1^{\{-1\}}$ satisfy the loop equation at leading order:

$$\begin{aligned}
(W_1^{\{-1\}}(x))^2 & = \frac{1}{(x-a_-)(x-a_+)} \\
& + \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \frac{1}{x-\xi} \frac{(\xi-a_-)(\xi-a_+)}{(x-a_-)(x-a_+)} (V^{\{0\}})'(\xi) W_1^{\{-1\}}(\xi).
\end{aligned} \tag{4-12}$$

Remark 4.1 Recall that $\text{supp } \rho = [\alpha_-, \alpha_+]$ is the discontinuity locus of $W_1^{\{-1\}}$. By the properties of the Stieltjes transform:

$$y(x) = \frac{(V^{\{0\}})'(x)}{2} - W_1^{\{-1\}}(x)$$

defines a holomorphic function on $U \setminus [\alpha_-, \alpha_+]$, and:

$$\forall x_0 \in [\alpha_-, \alpha_+], \quad \lim_{\epsilon \rightarrow 0^+} y(x_0 + i\epsilon) = i\pi \rho(x_0).$$

We state that there exists $M(x)$, continuous in some open neighborhood of $[\alpha_-, \alpha_+]$, such that:

$$y(x) = \frac{M(x)}{\sqrt{(x-\alpha_-)(x-\alpha_+)}}. \tag{4-13}$$

Proof. In Eqn. 4-12, we may first deform the contour $\mathcal{C}([a_-, a_+])$ to $\mathcal{C}([\alpha_-, \alpha_+])$. Secondly, we can rewrite:

$$\begin{aligned}
& (W_1^{\{-1\}}(x))^2 - (V^{\{0\}})'(x)W_1^{\{-1\}}(x) + \frac{U(x)}{(x-a_-)(x-a_+)} = 0, \\
U(x) & = -1 + \oint_{\mathcal{C}([\alpha_-, \alpha_+]) \cup \mathcal{C}(x)} \frac{d\xi}{2i\pi} \frac{(\xi-a_-)(\xi-a_+)}{x-\xi} (V^{\{0\}})'(\xi) W_1^{\{-1\}}(\xi),
\end{aligned}$$

where $U(x)$ is now holomorphic in some open neighborhood of $[\alpha_-, \alpha_+]$. So:

$$y(x) = \sqrt{\frac{R(x)}{(x - a_-)(x - a_+)}} , \quad R(x) = \frac{1}{4}(x - a_-)(x - a_+)(V^{\{0\}})'(x) - U(x).$$

This equation tells us that the discontinuity of y is of squareroot type. If $\alpha_- = a_-$ and $\alpha_+ = a_+$, we have Eqn. 4-13. If say $a_- < \alpha_-$, the fact that $y(x)$ has no discontinuity on $[a_-, \alpha_-[$ but a discontinuity on $[\alpha_-, \alpha_+]$ forces $R(x)$ to have a simple zero at $x = a_-$ and at $x = \alpha_-$, so that $y(x)$ is finite when $x = a_-$ and vanishes as $O(\sqrt{x - \alpha_-})$ when $x \rightarrow \alpha_-$. A similar statement holds if $a_+ > \alpha_+$. Then, Eqn. 4-13 holds a fortiori. \diamond

4.6 First correction to W_1

Let us reconsider Eqn. 4-11 (or the equivalent relation taking Remark 3.3 into account) after removing the 2nd and the 3rd line which has just been identified as the leading order. We can write as in § 4.3:

$$\left[\mathcal{K} + \widetilde{\delta\mathcal{K}} + \frac{1}{N} \left(1 - \frac{2}{\beta} \right) \frac{d}{dx} \right] \delta_{-1} W_1(x) = A_2 + C_0 + D_0,$$

where:

$$\begin{aligned} \widetilde{\delta\mathcal{K}}[f](x) &= -\mathcal{N}_{(\delta_0 V)'}[f(x)] + \delta_{-1} W_1(x) f(x), \\ A_2 &= -\frac{1}{N^2} W_2(x, x), \\ C_0 &= -\frac{1}{N} \left(1 - \frac{2}{\beta} \right) \left(\sum_{\tau \in \text{Hard}} \frac{1}{a_\tau - a_{-\tau}} \frac{1}{x - a_\tau} \right), \\ D_0 &= \sum_{\tau \in \text{Soft}} \frac{\partial_{a_\tau} \ln Z}{x - a_\tau}. \end{aligned}$$

By an argument similar to Eqn. 4-8, knowing that $W_2 \in O_l(N)$ implies that $\delta_{-1} W_1 \in O_{l+1}(1/N)$. Assuming further $W_3 \in O_{l'}(N)$ implies after Section 4.3 that $W_2 \in O_{l'+2}(1)$, so the 1st line of Eqn. 4-11 is subleading compared to the 3rd line. These two bounds are provided by Section 4.4 (the values of l and l' do not matter here). Hence:

Lemma 4.7 *There exists $W_1^{\{0\}} \in \mathcal{H}_{1;[\alpha_-, \alpha_+]}^{(1)}$ such that $W_1 = N W_1^{\{-1\}} + W_1^{\{0\}} + o(1)$. Explicitly:*

$$W_1^{\{0\}}(x) = \mathcal{K}^{-1} \left\{ - \left(1 - \frac{2}{\beta} \right) \left[\frac{d}{dx} (W_1^{\{-1\}}(x)) + \sum_{\tau \in \text{Hard}} \frac{1}{a_\tau - a_{-\tau}} \frac{1}{x - a_\tau} \right] + \mathcal{N}_{(V^{\{1\}})'}(W_1^{\{-1\}}(x)) \right\}.$$

This order was also obtained by [KS10] with similar arguments.

4.7 Recursion hypothesis at order k_0

Let $k_0 \geq -1$. We assume that the correlators W_n (for all $n \geq 1$) are determined up to a $o(N^{-k_0})$ for the norm $\|\cdot\|_{\Gamma_{l(k_0;n)}}$.

$$W_n(x_1, \dots, x_n) = \sum_{k=n-2}^{k_0} N^{-k} W_n^{\{k\}}(x_1, \dots, x_n) + N^{-k_0} \delta_{k_0} W_n(x_1, \dots, x_n). \quad (4-14)$$

Here, $W_n^{\{k\}}(x_1, \dots, x_n)$ are already known (they depend on β but not on N), and we call:

$$\omega_n^{\{k\}} = \sup_{\substack{1 \leq n' \leq n \\ -1 \leq k' \leq k}} \|W_n^{\{k\}}\|_{\Gamma_{l(k_0;n)}},$$

a bound for their norm. We can always assume that $l(k, n)$ defined for $-1 \leq k \leq k_0$ and $n \geq 1$ is an increasing function of k and n . Though the errors $\delta_{k_0} W_n$ are not supposed to be known, we assume that they are small:

$$\forall n \geq 1 \quad \|\delta_{k_0} W_n\|_{\Gamma_{l(k_0;n)}} \leq \epsilon_N^{\{k_0\}} \Delta_n^{\{k_0\}}.$$

Here, $\epsilon_N^{\{k_0\}}$ depends only on N and k_0 , and $\epsilon_N^{\{k_0\}} \rightarrow 0$ when $N \rightarrow \infty$, and $\Delta_n^{\{k_0\}}$ is a constant independent of N . We may assume that $\Delta_n^{\{k_0\}}$ increases with $n \geq 1$, upon replacement by $\sup_{1 \leq n' \leq n} \Delta_{n'}^{\{k_0\}}$. When $n > k_0 + 2$, we assume that Eqn. 4-14 reduces to:

$$W_n = N^{-k_0} \delta_{k_0} W_n.$$

Lemma 4.4 and Section 4.7 ensure that the initial ($k_0 = -1$) recursion hypothesis is satisfied. Moreover, we can take $\epsilon_N^{\{-1\}} = 1/N$, and up to a redefinition $\Gamma_k \rightarrow \Gamma_{k-m}$ for some integer m , we can take $l(-1; n) = 4(n-1)$.

4.8 Determination of $\delta_{k_0} W_{n_0}$

Let $n_0 \geq 1$. We now turn to the determination of the leading order of $\delta_{k_0} W_{n_0}(x, x_I)$. The case $(n_0, k_0) = (1, -1)$ is a bit special (because of the second term of the second line in Eqn. 4-11) and is given by Lemma 4.7. In all other cases, we consider the loop equation at rank n_0 (Thm. 3.2). Up to $o(N^{-(k_0-1)})$, the equation is true and involves quantities which are already known from the recursion hypothesis. The equality of the $o(N^{-(k_0-1)})$ involves the unknown $\delta_{k_0} W_{n_0}(x, x_I)$. The operator \mathcal{K} introduced in § 4.2.1 plays a special role. When the potential V has a $1/N$ expansion, the operator \mathcal{N} introduced in § 4.2.3 also appears, and we denote:

$$V = \sum_{k=0}^{k_0+1} N^{-k} V^{\{k\}} + N^{-(k_0+1)} \delta_{k_0+1} V.$$

We find:

$$N^{-(k_0-1)} \mathcal{K}(\delta_{k_0} W_{n_0})(x, x_I) = -N^{-k_0} E_{n_0}^{\{k_0\}}(x, x_I) - N^{-k_0} R_{n_0}^{\{k_0\}}(x, x_I), \quad (4-15)$$

with

$$\begin{aligned} E_{n_0}^{\{k_0\}}(x, x_I) &:= W_{n_0+1}^{\{k_0\}}(x, x, x_I) - \sum_{k=1}^{k_0+1} \mathcal{N}_{(V^{\{k\}})'}, [W_{n_0}^{\{k_0+1-k\}}(x, x_I)] \\ &+ \sum_{J \subseteq I} \sum_{k=0}^{k_0} W_{|J|+1}^{\{k\}}(x, x_J) W_{n_0-|J|}^{\{k_0-k\}}(x, x_{I \setminus J}) + \left(1 - \frac{2}{\beta}\right) \frac{d}{dx} \left(W_{n_0}^{\{k_0\}}(x, x_I)\right) \\ &+ \frac{2}{\beta} \sum_{i \in I} \frac{d}{dx_i} \left\{ \frac{W_{n_0-1}^{\{k_0\}}(x, x_{I \setminus \{i\}})}{x - x_i} - \frac{L(x_i)}{L(x)} \left(\frac{1}{x - x_i} + c\right) W_{n_0-1}^{\{k_0\}}(x_I) \right\}, \end{aligned}$$

and the remaining

$$\begin{aligned} R_{n_0}^{\{k_0\}}(x, x_I) &:= \delta_{k_0} W_{n_0+1}(x, x, x_I) + \sum_{k=1}^{k_0} N^{-k} \sum_{k'=0}^{k_0} \sum_{J \subseteq I} W_{|J|+1}^{\{k'\}}(x, x_J) W_{n_0-|J|}^{\{k_0+k-k'\}}(x, x_{I \setminus J}) \\ &+ \sum_{k=0}^{k_0} N^{-k} \sum_{J \subseteq I} (\delta_{k_0} W_{|J|+1})(x, x_J) W_{n_0-|J|}^{\{k\}}(x, x_{I \setminus J}) \\ &+ N^{-k_0} \sum_{J \subseteq I} (\delta_{k_0} W_{|J|+1})(x, x_J) (\delta_{k_0} W_{n_0-|J|})(x, x_{I \setminus J}) \\ &+ \left(1 - \frac{2}{\beta}\right) \frac{d}{dx} \left((\delta_{k_0} W_{n_0})(x, x_I) \right) - (\delta_{k_0} W_{n_0+1})(x, x, x_I) \\ &- \sum_{k=0}^{k_0} N^{-k} \mathcal{N}_{(\delta_{k_0+1} V)'}, [W_{n_0}^{\{k\}}(x, x_I)] - \sum_{k=0}^{k_0} N^{-k} \mathcal{N}_{(V^{\{k+1\}})'}, [(\delta_{k_0} W_{n_0})(x, x_I)] \\ &- N^{-k_0} \mathcal{N}_{(\delta_{k_0+1} V)'}, [(\delta_{k_0} W_{n_0})(x, x_I)] \\ &+ \frac{2}{\beta} \sum_{i \in I} \frac{d}{dx_i} \left\{ \frac{(\delta_{k_0} W_{n_0-1})(x, x_{I \setminus \{i\}})}{x - x_i} - \frac{L(x_i)}{L(x)} \left(\frac{1}{x - x_i} + c\right) (\delta_{k_0} W_{n_0-1})(x_I) \right\} \\ &+ \frac{2}{\beta} \sum_{\tau \in \text{Soft}} N^{k_0} \frac{\partial_{a_\tau} W_{n-1}(x_I)}{x - a_\tau}. \end{aligned} \quad (4-16)$$

It is understood that \mathcal{K} and \mathcal{N}_g operate on the x variable. The variables x_I are spectators. Notice that this equation is linear in $\delta_{k_0} W_{n_0}$, up to a small quadratic term.

Looking naively at this equation, we see that the leading term of $\delta_{k_0} W_{n_0}$ happens to be of order $1/N$ (giving a $N^{-(k_0+1)}$ contribution to W_{n_0}), and is obtained by applying \mathcal{K}^{-1} to $E_{n_0}^{\{k_0\}}(x, x_I)$. To make this idea rigorous, let us bound $R_{n_0}^{\{k_0\}}$. Even if some terms in the right hand side have not been determined yet (like $\delta_{k_0} W_n$ that we are just considering), we already know a bound for each of them from the recursion hypothesis.

Very rough bounds are enough, we just need to show that the right hand side is small when $N \rightarrow \infty$. When $k_0 = -1$, we must pay special attention at the terms involving N^{-k_0} directly, i.e. the 3rd line and the 6th line in Eqn. 4-16. In the 6th line, $(\delta_{k_0+1}V)'$ is of order N^{-1} , so we obtain a term of order $\epsilon_N^{\{k_0\}}$, which is always small. The 3rd line is of order $N(\epsilon_N^{\{-1\}})^2$, which is also small since we have here $\epsilon_N^{\{-1\}} = 1/N$ (Lemma 4.7). For N large enough, we have:

$$\begin{aligned}
\|R_{n_0}^{\{k_0\}}\|_{\Gamma_{l(k_0+1;n_0)}} &\leq \epsilon_N^{\{k_0\}} \Delta_{n_0+1}^{\{k_0\}} + N^{-1} (k_0 + 1) 2^{n_0-1} (\omega_{n_0}^{\{k_0\}})^2 \\
&\quad + \epsilon_N^{\{k_0\}} 2^{n_0-1} \Delta_{n_0}^{\{k_0\}} \omega_{n_0}^{\{k_0\}} + (\epsilon_N^{\{k_0\}})^2 N^{-k_0} (\Delta_{n_0}^{\{k_0\}})^2 \\
&\quad + \epsilon_N^{\{k_0\}} \left| 1 - \frac{2}{\beta} \right| \zeta_{l(k_0;n_0)} \Delta_{n_0}^{\{k_0\}} + N^{-1} \sum_{k=0}^{k_0} \|\mathcal{N}_{(\delta_{k_0+1}V)'}\|_{\Gamma_{l(k_0;n_0)}} \Delta_{n_0}^{\{k_0\}} \\
&\quad + \epsilon_N^{\{k_0\}} \sum_{k=0}^{k_0} \|\mathcal{N}_{(V^{\{k\}})'}\|_{\Gamma_{l(k_0;n_0)}} \Delta_{n_0}^{\{k_0\}} + \epsilon_N^{\{k_0\}} N^{-k_0} \|\mathcal{N}_{(\delta_{k_0+1}V)'}\|_{\Gamma_{l(k_0;n_0)}} \Delta_{n_0}^{\{k_0\}} \\
&\quad + \epsilon_N^{\{k_0\}} \frac{2}{\beta} \zeta_{l(k_0;n_0-1)} \left(|c| + \zeta_{l(k_0;n_0-1)+1} \frac{\sup_{\xi \in \Gamma_{l(k_0;n_0-1)}} |L(\xi)|}{\inf_{x \in \Gamma_{l(k_0;n_0-1)}} |L(x)|} \right) \Delta_{n_0-1}^{\{k_0\}} \\
&\quad + \frac{2}{\beta} \frac{2\gamma_n}{d(\Gamma, [a_-, b_-])^n} N^{k_0+n-1} e^{-N\eta}
\end{aligned}$$

Given the control provided by the recursion hypothesis, this inequality is correct provided we choose:

$$l(k_0 + 1; n_0) \geq \max[l(k_0; n_0 - 1) + 2, l(k_0; n_0) + 1, l(k_0; n_0 + 1)]. \quad (4-17)$$

Accordingly, $R_{n_0}^{\{k_0\}} \rightarrow 0$ when $N \rightarrow \infty$. Eqn. 4-15 tells us that $E_{n_0}^{\{k_0\}} + R_{n_0}^{\{k_0\}} \in \text{Im } \mathcal{K}$ for any N . Since $\text{Im } \mathcal{K}$ is closed (Lemma 4.1), we know that $E_{n_0}^{\{k_0\}} \in \text{Im } \mathcal{K}$, and also by difference $R_{n_0}^{\{k_0\}} \in \text{Im } \mathcal{K}$ for any N . And, by continuity of \mathcal{K}^{-1} , we deduce:

$$\delta_{k_0} W_{n_0} = \frac{1}{N} W_{n_0}^{\{k_0+1\}} + \frac{1}{N} \delta_{k_0+1} W_{n_0},$$

where:

$$W_{n_0}^{\{k_0+1\}} = -\mathcal{K}^{-1}[E_{n_0}^{\{k_0\}}], \quad \delta_{k_0+1} W_{n_0} = -\mathcal{K}^{-1}[R_{n_0}^{\{k_0\}}] \in o(1). \quad (4-18)$$

The previous inequality is more precise about the $o_{l(k_0+1;n_0)}(1)$: there exists a constant $\Delta_{n_0}^{\{k_0+1\}}$, such that

$$\|\delta_{k_0+1} W_{n_0}\|_{\Gamma_{l(k_0+1;n_0)}} \leq \Delta_{n_0}^{\{k_0+1\}} \max(N^{-1}; \epsilon_N^{\{k_0\}}).$$

4.9 Remarks

The recursion hypothesis tells us that $W_{n_0}^{\{k_0\}} = 0$ whenever $n > k_0 + 2$ (we call $\star[k_0]$ this recursive assumption). Let us see what happens at order $k_0 + 1$ (here, k_0 is fixed, but n_0 is free), by looking at Eqn. 4-18.

- The term $W_{n_0+1}^{\{k_0\}}$ vanishes whenever $n_0 > k_0 + 1$.
- The term $W_{|J|+1}^{\{k\}} W_{n_0-|J|}^{\{k_0-k\}}$ may be non zero in case $k + 1 \geq |J| \geq n_0 - k_0 - 2 + k$. This is impossible to fulfil as soon as $n_0 > k_0 + 3$.
- The term $\left(W_{n_0}^{\{k_0\}}\right)'$ vanishes whenever $n_0 > k_0 + 2$.
- The term involving $W_{n_0-1}^{\{k_0\}}$ vanishes whenever $n_0 > k_0 + 3$.

Accordingly, $W_{n_0}^{\{k_0+1\}} \equiv 0$ when $n_0 > (k_0 + 1) + 2$, i.e. $\star[k_0 + 1]$ holds. This is just the manifestation of Lemma 4.2. Hence, we have propagated the full recursion hypothesis to order $k_0 + 1$. An easy recursion shows that $W_n^{\{k\}}$ are actually holomorphic functions on the domain $\mathbb{C} \setminus [\alpha_-, \alpha_+]$, i.e belongs to the subspace $\mathcal{H}_{n;[\alpha_-, \alpha_+]}^{(1)}$ of $\mathcal{H}_{n;[a_-, a_+]}^{(1)}$. Therefore, we can contract the contour to $\mathcal{C}([\alpha_-, \alpha_+])$ in the expression of \mathcal{K}^{-1} (Eqn. 4-5) when computing $W_n^{\{k\}}$ with formula 4-18.

Since $l(k; n) = 4(n - 1)$, the minimal solution of Eqn. 4-17 is $l(k; n) = 4(n + k)$. Indeed, in this proof, we need to have a more restrictive control on the error done at height $n + k$, in order to bound the error done at height $n + k + 1$. Nevertheless, since $\Gamma_l \subseteq \text{Int}(\Gamma_E)$ for all l , we can at the end make the weaker statement that, for any n and k :

$$\|\delta_k W_n\|_{\Gamma_E} \rightarrow 0 \quad (4-19)$$

when $N \rightarrow \infty$. However, we necessarily have $d(\Gamma_l, \Gamma_{l+1}) \rightarrow 0$ when $l \rightarrow \infty$, so that the constant ζ_l which allows us to bound the derivative of a function with the function itself (Eqn. 4-3), blows up. This means that Eqn. 4-19 cannot be uniform³ in n and k , even when $\beta = 2$.

A posteriori, from Eqn. 4-19, we can deduce by choosing rather $l(k_0; n_0) = 8(n_0 + k_0)$:

$$\|\delta_{k_0} W_{n_0}\| \leq N^{-1} \|W_{n_0}^{\{k_0+1\}}\|_{\Gamma_{l(k_0+1; n_0)}} + \Delta_{n_0}^{\{k_0\}} \max(N^{-1}; \epsilon_N^{\{k_0\}}).$$

Subsequently, upon redefinition of the constant $\Delta_{n_0}^{\{k_0\}}$, we may choose $\epsilon_N^{\{k_0\}} = 1/N$. Finally, we can make the weaker statement that, for any n and k :

$$\|\delta_k W_n\|_{\Gamma_E} \in o(1/N),$$

without uniformity in n and k .

³We thank Pavel Bleher for pointing out a mistake in a former version of the article, which we corrected by introducing this family of nested contours.

5 Proof of the main results

5.1 Expansion of the correlators

We wish to study the β ensembles on a given interval $[b_-, b_+]$, with the hypotheses 1.1 on the potential V . When both edges are hard, Hyp. 1.1 are equivalent to the five assumptions of Section 4, so the Proposition 1.1 is already proved, as we have shown recursively that Eqn. 4-14 holds for all k_0 . Let us now assume that one of the edge is soft. The equilibrium measure $\mu_{\text{eq}} := \mu_{\text{eq}}^{V;[b_-, b_+]}$ with support $[\alpha_-, \alpha_+] \subset [b_-, b_+]$ also coincides with $\mu_{\text{eq}}^{V;[a_-, a_+]}$, where a_- can be any point in $[b_-, \alpha_-[$ if b_- is a soft edge, and $a_- = b_-$ else (resp. a_+ can be any point in $]\alpha_+, b_+]$ if b_+ is a soft edge, and $a_+ = b_+$ else). When b_τ is a soft edge, "offcriticality" implies that $S(x)$ is positive in a neighborhood of α_τ in $[b_-, b_+] \setminus [\alpha_-, \alpha_+]$. So, one can choose an interval $[a_-, a_+] \subseteq U$, and such that the five assumptions of Section 4 are satisfied for $d\mu_{N, \beta}^{V;[a_-, a_+]}$. Theorem 4.1 then can be applied: there exists an asymptotic expansion

$$W_n^{V;[a_-, a_+]}(x_1, \dots, x_n) = \sum_{k \geq n-2} N^{-k} W_n^{V;\{k\}}(x_1, \dots, x_n), \quad (5-1)$$

with respect to the norm $\|\cdot\|_{\Gamma_E}$ where $\Gamma_E \subseteq U$ can be any contour surrounding $[a_-, a_+]$ but not the zeroes of S . The "large deviation control" on $[b_-, b_+]$ allows to use Proposition 2.2: there exists $\eta > 0$ such that, for any contour $\Gamma'_E \subseteq \mathbb{C}$ surrounding $[b_-, b_+]$, there exists $T_{n, \Gamma} > 0$ such that:

$$\|W_n^{V;[b_-, b_+]} - W_n^{V;[a_-, a_+]}\|_{\Gamma'_E} \leq T_{n, \Gamma'_E} e^{-N\eta}.$$

This implies that the right hand side of Eqn. 5-1 is an asymptotic series for $W_n^{V;[b_-, b_+]}(x_1, \dots, x_n)$, uniformly for (x_1, \dots, x_n) in any compact of $(\mathbb{C} \setminus [b_-, b_+])^n$.

We give below a more transparent condition, which imply the "large deviation control" assumption on $[b_-, b_+]$:

Remark 5.1 *If $S(x) > 0$ whenever $x \in [b_-, b_+]$, then $\mathcal{J}^{V;[b_-, b_+]}$ achieves its minimum value only on $[\alpha_-, \alpha_+]$,*

Indeed, $\mathcal{J}^{V;[b_-, b_+]}(x)$ is differentiable when $x \in]b_-, b_+[\setminus [\alpha_-, \alpha_+]$, and we have:

$$(\mathcal{J}^{V;[b_-, b_+]}(x))' = \frac{(V^{\{0\}})'(x)}{2} - W_1^{\{-1\}}(x) = y(x) = S(x)\sigma(x).$$

The sign of the square root $\sigma(x)$ is determined for example by the positivity conditions 1-2 on $\mathcal{J}^{V;[b_-, b_+]}$. If we assume that S do not vanish on $[b_-, b_+]$, this implies that $\mathcal{J}^{V;[b_-, b_+]}$ is strictly decreasing in $[b_-, \alpha_-[$ and strictly increasing on $]\alpha_+, b_+]$, hence the remark.

5.2 Expansion of the free energy

So far, we only have determined the expansion of the correlators which are by definition derivatives of the free energy. To find the free energy itself, one would like to interpolate between our initial potential V , and a simpler situation, using that the difference depends on the correlators. For any fixed $\alpha_- < \alpha_+$, and fixed nature of the edges $X_{\pm} \in \{\text{hard}, \text{soft}\}$, we denote by $\mathcal{V}_{\alpha_-, X_-}^{\alpha_+, X_+}$ the set of potentials V :

- defined at least on some interval $[a_-, a_+] \supseteq [\alpha_-, \alpha_+]$, with $a_{\tau} \neq \alpha_{\tau}$ if $X_{\tau} = \text{soft}$, and $a_{\tau} = \alpha_{\tau}$ if $X_{\tau} = \text{hard}$;
- which satisfies the five assumptions of Section 4.1 on $[a_-, a_+]$, in particular is offcritical on $[a_-, a_+]$;
- for which the equilibrium measure $\mu_{\text{eq}}^{V; [a_-, a_+]}$ has $[\alpha_-, \alpha_+]$ as support,
- and such that a_{τ} is an edge of nature X_{τ} .

Lemma 5.1 $\mathcal{V}_{\alpha_-, X_-}^{\alpha_+, X_+}$ is a convex set.

Proof. Let $V_0, V_1 \in \mathcal{V}_{\alpha_-, X_-}^{\alpha_+, X_+}$, and set $V_s = (1-s)V_0 + sV_1$ for $s \in [0, 1]$. V_0 and V_1 are at least defined on a common interval $[a_-, a_+] \supseteq [\alpha_-, \alpha_+]$. Let us call $\nu_s = d\mu_{\text{eq}}^{V_s; [a_-, a_+]}$ the equilibrium measure for the potential V_s on $[a_-, a_+]$. We observe that $(1-s)d\nu_0 + sd\nu_1$ is a probability measure which is solution of the characterization of dL_s by Thm. 1.1. Therefore, $dL_s = (1-s)dL_0 + sdL_1$. Besides, we know that there exists a function S_s , regular in a neighborhood of $[\alpha_-, \alpha_+]$ in the complex plane, positive on $[a_-, a_+]$, such that:

$$d\nu_s(\xi) = \frac{d\xi}{\pi} S_s(\xi) \sqrt{\frac{\prod_{\tau/X_{\tau}=\text{soft}} |\xi - \alpha_{\tau}|}{\prod_{\tau'/X_{\tau'}=\text{hard}} |\xi - \alpha_{\tau'}|}} \mathbf{1}_{[\alpha_-, \alpha_+]}(\xi),$$

for $s = 0$ or $s = 1$. Since the edges are of the same nature in V_0 et V_1 , we must have $S_s = (1-s)S_0 + sS_1$. Since S_0 and S_1 are positive on $[a_-, a_+]$, so is S_s . Hence $V_s \in \mathcal{V}_{\alpha_-, X_-}^{\alpha_+, X_+}$.

Corollary 5.1 Let $V_0, V_1 \in \mathcal{V}_{\alpha_-, X_-}^{\alpha_+, X_+}$. When a_- and a_+ satisfy the condition above, the quantity:

$$\ln Z_{N, \beta}^{V_1; [a_-, a_+]} - \ln Z_{N, \beta}^{V_0; [a_-, a_+]} = -\frac{N\beta}{2} \int_0^1 ds \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} (V_1(\xi) - V_0(\xi)) W_1^{V_s; [a_-, a_+]}(\xi)$$

has a large N asymptotic expansion of the form:

$$\ln Z_{N, \beta}^{V_1} - \ln Z_{N, \beta}^{V_0} = \sum_{k \geq -2} N^{-k} F_{\beta}^{V_0 \rightarrow V_1; [a_-, a_+]; \{k\}},$$

where:

$$F_\beta^{V_0 \rightarrow V_1; [a_-, a_+]; \{k\}} = -\frac{\beta}{2} \int_0^1 ds \oint_{\mathcal{C}([a_-, a_+])} \frac{d\xi}{2i\pi} \sum_{m=0}^{k+2} (V_1^{\{m\}}(\xi) - V_0^{\{m\}}(\xi)) (W_1^{V_s; [a_-, a_+]})^{\{k+1-m\}}(\xi).$$

Proof. Since V_s satisfies the five assumptions of Section 4.1 for any $s \in [0, 1]$, we can apply our main theorem to $W_1^{V_s}$. Moreover, since we do not reach a critical point when s is in the compact $[0, 1]$, we know that the error $O(N^{-K})$ made if we replace $W_1^{V_s}$ by $\sum_{k=-1}^{K-1} N^{-k} W_1^{V_s; \{k\}}$ is uniformly bounded with respect to s on some contour surrounding $[a_-, a_+]$ and in the analyticity domain of V . Therefore, we can exchange the integral and the sum in the asymptotic expansion. \diamond

For instance, when V satisfies the five assumptions of Section 4.1 on some interval $[a_-, a_+]$, such that a_\pm are soft edges, one can interpolate between V and a Gaussian potential corresponding to an equilibrium measure with support $[\alpha_-, \alpha_+]$:

$$V_{G, \alpha_-, \alpha_+}(x) = \frac{8}{(\alpha_+ - \alpha_-)^2} \left(x - \frac{\alpha_- + \alpha_+}{2} \right)^2.$$

Proposition 5.1 *Let V be a potential satisfying the five assumptions of Section 4.1 on some interval $[a_-, a_+]$, such that a_\pm are soft edges. For all $s \in [0, 1]$, $(1-s)V + sV_{G, \alpha_-, \alpha_+}$ belongs to $\mathcal{V}_{\alpha_-, \text{soft}}^{\alpha_+, \text{soft}}$ and we have the following asymptotic expansion when $N \rightarrow \infty$:*

$$Z_{N, \beta}^V = Z_{N, \beta}^{V_{G, \alpha_-, \alpha_+}} \exp \left(\sum_{k \geq -2} N^{-k} F_\beta^{V \rightarrow V_{G, \alpha_-, \alpha_+}; [a_-, a_+]; \{k\}} \right),$$

where the prefactor is a partition function of the Gaussian β ensemble (see Eqn. 1-4):

$$Z_{N, \beta}^{V_{G, \alpha_-, \alpha_+}} = Z_{N, G\beta E} \left(\frac{\alpha_+ - \alpha_-}{4} \right)^{N + \frac{\beta}{2} N(N-1)}.$$

According to the discussion of § 5.1, we can weaken the hypothesis of the proposition above to find Theorem 1.2.

5.3 Central limit theorem

Eventually, our results imply the central limit theorem proved by Johansson [Joh98], but here integration is taken on a compact set $[a_-, a_+]$ instead of the real line (in fact as our derivation is quite similar to Johansson's, this is not surprising). For simplicity, we take here the hypotheses of Section 4, although we could refine to hypotheses 1.1 following § 5.1.

Let $h : [a_-, a_+] \rightarrow \mathbb{R}$ be a function which can be extended as a holomorphic function defined on some neighborhood of $[a_-, a_+]$, let us take $V \equiv V^{\{0\}}$ independent

of N , $V^{\{1\}} = \frac{2}{\beta} h$ and define $V_h = V^{\{0\}} + N^{-1} V^{\{1\}} = V - \frac{2}{N\beta} h$. Then:

$$\mu_{N,\beta}^{V;[a_-,a_+]} \left[\exp \left(\sum_{i=1}^N h(\lambda_i) \right) \right] = \frac{Z_{N,\beta}^{V_h;[a_-,a_+]}}{Z_{N,\beta}^{V;[a_-,a_+]}} ,$$

and we can use Corollary 5.1 to derive its large N asymptotics. Indeed, we have

$$\ln \mu_{N,\beta}^{V;[a_-,a_+]} \left[\exp \left(\sum_{i=1}^N h(\lambda_i) \right) \right] = \int_0^1 ds \oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi} W_1^{V_{sh}}(\xi) h(\xi).$$

By Theorem 5.1, or simply at the point of Lemma 4.7, we have:

$$\begin{aligned} W_1^{V_{sh};\{-1\}}(\xi) &= W_1^{V;\{-1\}}(\xi) = \int \frac{d\mu_{\text{eq}}(\eta)}{\xi - \eta}, \\ W_1^{V_{sh};\{0\}}(\xi) &= \mathcal{K}^{-1} \left\{ - \left(1 - \frac{2}{\beta} \right) \left[\frac{d}{dx} (W_1^{V;\{-1\}}(x)) + \sum_{\tau \in \text{Hard}} \frac{1}{a_\tau - a_{-\tau}} \frac{1}{x - a_\tau} \right] \right. \\ &\quad \left. - \frac{2s}{\beta} \mathcal{N}_{h'}(W_1^{V;\{-1\}})(x) \right\}, \\ W_1^{V_{sh}} &= N W_1^{V_{sh};\{-1\}} + W_1^{V_{sh};\{0\}} + o(1), \end{aligned}$$

which shows the:

Proposition 5.2 *Central limit theorem.*

$$\ln \mu_{N,\beta}^{V;[a_-,a_+]} \left[\exp \left(\sum_{i=1}^N h(\lambda_i) \right) \right] = N \int d\mu_{\text{eq}}(\eta) h(\eta) + m[h] + \frac{1}{2} C[h] + o(1),$$

with $m[h]$ the linear in the function h , given by:

$$m[h] = - \left(1 - \frac{2}{\beta} \right) \oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi} \mathcal{K}^{-1} \left\{ \frac{d}{dx} (W_1^{V;\{-1\}}(x)) + \sum_{\tau \in \text{Hard}} \frac{1}{a_\tau - a_{-\tau}} \frac{1}{x - a_\tau} \right\} h(\xi),$$

and $C[h]$ the quadratic function of h given by:

$$C[h] = - \frac{2}{\beta} \oint_{\mathcal{C}([a_-,a_+])} \frac{d\xi}{2i\pi} \mathcal{K}^{-1} \left[\mathcal{N}_{h'}(W_1^{V;\{-1\}}) \right](\xi) h(\xi).$$

Therefore $\sum_{i=1}^N h(\lambda_i) - N \int d\mu_{\text{eq}}(\eta) h(\eta)$ converges towards a Gaussian variable with mean $m[h]$ and covariance $C[h]$.

A Proof of Proposition 2.1

We use the notation introduced in Proposition 2.1, in particular the eigenvalues are integrated over a segment $[b_-, b_+]$ which may not be compact.

A.1 $\tilde{\mathcal{J}}^{V;[b_-,b_+]}$ is a good rate function

$\tilde{\mathcal{J}}^{V;[b_-,b_+]}$ is lower semicontinuous as a supremum of the continuous functions

$$\tilde{\mathcal{J}}_\varepsilon^{V;[b_-,b_+]}(x) := \frac{V(x)}{2} - \int_{b_-}^{b_+} d\mu_{\text{eq}}^{V;[b_-,b_+]}(\xi) \ln [\max(|x - \xi|, \varepsilon)] - \inf_{\xi \in [b_-,b_+]} \mathcal{J}^{V;[b_-,b_+]}(\xi).$$

Moreover, by the assumption of Eqn. 2-1, it goes to infinity at infinity. Hence, $\tilde{\mathcal{J}}^{V;[b_-,b_+]}$ has compact level sets. Since it is non-negative, it is a good rate function.

A.2 The law of the extreme eigenvalues is exponentially tight

Exponential tightness of the extreme eigenvalues means:

$$\limsup_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;[b_-,b_+]}(\lambda_{\max} \geq M \text{ or } \lambda_{\min} \leq -M) = -\infty. \quad (1-1)$$

By [AGZ10, Lemma 2.6.7], it is enough to show that:

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \frac{Z_{N-1,\beta}^{V;[b_-,b_+]}}{Z_{N,\beta}^{V;[b_-,b_+]}} < \infty. \quad (1-2)$$

For this purpose, observe that by Jensen's inequality

$$\begin{aligned} \frac{Z_{N,\beta}^{V;[b_-,b_+]}}{Z_{N-1,\beta}^{V;[b_-,b_+]}} &= \mu_{N-1,\beta}^{V;[b_-,b_+]} \left[\int_{b_-}^{b_+} d\lambda_N \exp \left(\beta \sum_{i=1}^{N-1} \ln |\lambda_N - \lambda_i| - \frac{\beta N}{2} V(\lambda_N) - \frac{\beta}{2} \sum_{i=1}^{N-1} V(\lambda_i) \right) \right] \\ &\geq \kappa \exp \left\{ \frac{\beta}{2} (\mu_{N-1,\beta}^{V;[b_-,b_+]} \otimes \chi) \left[2 \sum_{i=1}^{N-1} \ln |\lambda_N - \lambda_i| - (N-1)V(\lambda_N) - \sum_{i=1}^{N-1} V(\lambda_i) \right] \right\}, \end{aligned}$$

where we denoted χ the law on λ_N given by:

$$d\chi(x) = \frac{\mathbf{1}_{[b_-,b_+]}(x) dx}{\kappa} e^{-\frac{\beta}{2} V(x)}, \quad \kappa = \int_{b_-}^{b_+} d\xi e^{-\frac{\beta}{2} V(\xi)}.$$

The function $\xi \mapsto \int_{\mathbb{R}} d\chi(\lambda_N) \ln |\lambda_N - \xi|$ is bounded on compact sets and going to infinity like $\ln |\xi|$, so is bounded from below, by a constant $\frac{\kappa_1}{2}$. We can rewrite:

$$\frac{Z_{N,\beta}^{V;[b_-,b_+]}}{Z_{N-1,\beta}^{V;[b_-,b_+]}} \geq \kappa \exp \left\{ \beta(N-1) \left[\kappa_1 - \chi[V] - \mu_{N-1,\beta}^{V;[b_-,b_+]}[L_{N-1}(V)] \right] \right\}.$$

By exponential tightness [AGZ10, Eqn. 2.6.21], we know that there exists a constant $\kappa_2 > 0$ so that

$$- \mu_{N-1,\beta}^{V;[b_-,b_+]}[L_{N-1}(V)] \geq - \mu_{N-1,\beta}^{V;[b_-,b_+]}[L_{N-1}(|V|)] \geq -\kappa_2.$$

So, if we set $\kappa_3 = \chi[V]$ and choose κ_2 large enough, we have:

$$\frac{Z_{N,\beta}^{V;[b_-,b_+]}}{Z_{N-1,\beta}^{V;[b_-,b_+]}} \geq \kappa e^{-\beta(N-1)\delta},$$

with a positive constant $\delta = -\kappa_1 + \kappa_2 + \kappa_3$. This justifies Eqn. 1-2 and completes the proof of Eqn. 1-1.

A.3 Upper bound for large deviation of the extreme eigenvalues

We give the argument for the minimal eigenvalue, the case of the maximal eigenvalue being similar. By exponential tightness (Eqn. 1-1), it is enough to prove a weak large deviation upper bound, that is control the probability of small balls. First, observe that for any $x - \alpha_- \geq 2\epsilon > 0$,

$$\mu_{N,\beta}^{V;[b_-,b_+]}\left[\lambda_{\min} \geq x\right] \leq \mu_{N,\beta}^{V;[b_-,b_+]}\left[L_N(\mathbf{1}_{[\alpha_-, \alpha_- + \epsilon]}) = 0\right]$$

is of order $e^{-N^2\kappa_\epsilon}$ for some $\kappa_\epsilon > 0$ by the large deviation principle for the law of L_N under $\mu_{N,\beta}^{V;[b_-,b_+]}$, see e.g. [AG97] or [AGZ10, Theorem 2.6.1]. Moreover, the probability that λ_{\min} is smaller than a_- vanishes and therefore we have

$$\limsup_{\epsilon \downarrow 0} \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;[b_-,b_+]}\left(\lambda_{\min} \in]-\infty, b_- - \epsilon] \cup [\alpha_- + \epsilon, +\infty[)\right) = -\infty.$$

Hence, we may and shall concentrate on probability of deviating on $[b_-, \alpha_-]$, and actually we may restrict ourselves to the case where b_- and b_+ are finite by Eqn. 1-1. We let F be a closed subset of $[b_-, \alpha_-]$. We then have:

$$\mu_{N,\beta}^{V;[b_-,b_+]}\left[\lambda_{\min} \in F\right] = Y_N \int_F d\xi e^{-\frac{\beta}{2}V(\xi)} \Xi_N(\xi),$$

where we introduced:

$$Y_N = \frac{Z_{N-1,\beta}^{\frac{N}{N-1}V;[b_-,b_+]}}{Z_{N,\beta}^{V;[b_-,b_+]}},$$

$$\Xi_N(\xi) = \mu_{N-1,\beta}^{\frac{N}{N-1}V;[b_-,b_+]}\left(e^{\beta \sum_{i=1}^{N-1} \ln|\xi - \lambda_i| - \frac{\beta}{2}(N-1)V(\xi)} \prod_{i=1}^{N-1} \mathbf{1}_{[b_-, \lambda_i]}(\xi)\right).$$

Upper bound for $\Xi_N(\xi)$

Notice that the logarithm is uniformly bounded from above on compacts so that the exponent is at most of order N . Therefore, we may and shall assume that under

$\mu_{N-1, \beta}^{\frac{N}{N-1}V; [b_-, b_+]}$, L_{N-1} is at a distance smaller than $\kappa > 0$ from the equilibrium measure $\mu_{\text{eq}} := \mu_{\text{eq}}^{V; [b_-, b_+]}$, since the opposite event has probability smaller than $e^{-\Gamma_\kappa(N-1)^2}$ for some $\Gamma_\kappa > 0$, see e.g. [AGZ10, Theorem 2.6.1]. Here, the distance can be taken to be any distance compatible with the weak topology, e.g. the Wasserstein distance. Thus, we have for large N :

$$\Xi_N(\xi) \leq e^{-\Gamma_\kappa N^2/2} + e^{\beta(N-1) \sup_{d(\mu, \mu_{\text{eq}}) < \kappa} \left(-\frac{V(\xi)}{2} + \int \ln |\xi - \eta| d\mu(\eta) \right)},$$

where we take the supremum over probability measures μ on $[b_-, b_+]$ with Wasserstein distance to μ_{eq} strictly smaller than κ . We observe also that for all probability measures μ on $[b_-, b_+]$, and for any $\zeta > 0$:

$$\int_{b_-}^{b_+} \ln |\xi - \eta| d\mu(\eta) \leq \phi_\zeta(\mu, \xi) = \int_{b_-}^{b_+} \ln [\max(|\xi - \eta|, \zeta)] d\mu(\eta)$$

where $\phi_\zeta(\mu, \xi)$ is continuous in μ and ξ , and $\phi_\zeta(\mu_{\text{eq}}, \xi)$ converges towards $\phi_0(\mu_{\text{eq}}, \xi)$ as ζ goes to zero. We deduce that:

$$\limsup_{\kappa \downarrow 0} \sup_{\xi \in F} \sup_{d(\mu, \mu_{\text{eq}}) < \kappa} \beta \left(\int \ln |\xi - \eta| d\mu(\eta) - \frac{V(\xi)}{2} \right) \leq -\beta \inf_{\xi \in F} \mathcal{J}^{V; [b_-, b_+] }(\xi).$$

Therefore, for any $\eta' > 0$, and N large enough, we conclude that:

$$\sup_{\xi \in F} \Xi_N(\xi) \leq e^{N(\eta' - \beta \inf_{\xi \in F} \mathcal{J}^{V; [b_-, b_+] }(\xi))}. \quad (1-3)$$

Lower bound for Y_N

We observe that, for any $\varepsilon > 0$ small enough, and any $x \in [b_- + \varepsilon, b_+ - \varepsilon]$, there exists δ_ε going to zero with ε so that

$$\begin{aligned} \frac{1}{Y_N} &= \frac{Z_{N, \beta}^{V; [b_-, b_+]}}{Z_{N-1, \beta}^{\frac{N}{N-1}V; [b_-, b_+]}} \\ &= \mu_{N-1, \beta}^{\frac{N}{N-1}V; [b_-, b_+] } \left(\int_{b_-}^{b_+} d\xi e^{-\frac{\beta N}{2}V(\xi)} \prod_{i=1}^{N-1} |\xi - \lambda_i|^\beta \right) \\ &\geq \mu_{N-1, \beta}^{\frac{N}{N-1}V; [b_-, b_+] } \left(\int_{x-\varepsilon}^{x+\varepsilon} d\xi e^{-\frac{\beta N}{2}V(\xi)} \prod_{i=1}^{N-1} |\xi - \lambda_i|^\beta \right) \\ &\geq 2\varepsilon e^{-\frac{\beta N}{2}V(x) - N\delta_\varepsilon} \mu_{N-1, \beta}^{\frac{N}{N-1}V; [b_-, b_+] } \left(e^{\sum_{i=1}^{N-1} \frac{\beta}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \ln |\xi - \lambda_i| d\xi} \right), \end{aligned}$$

where we have finally used Jensen's inequality. But $\lambda \rightarrow \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \ln |\xi - \lambda| d\xi$ is bounded continuous on $[a_-, a_+]$ and therefore by the large deviation principle for the law of the

empirical measure L_{N-1} under $\mu_{N-1,\beta}^{V;[b_-,b_+]}$ (with rate function which vanishes only at μ_{eq}) we deduce that for N large enough:

$$\frac{1}{Y_N} \geq 2\varepsilon e^{-\frac{\beta N}{2}V(x)-2N\delta_\varepsilon} e^{(N-1) \int \frac{\beta}{2\varepsilon} \left(\int_{x-\varepsilon}^{x+\varepsilon} \ln|\xi-\lambda|d\xi \right) d\mu_{\text{eq}}(\lambda)}.$$

Hence, by taking ε sufficiently small independently of N , and optimizing over the choice of $x \in]b_-, b_+[$, we conclude that for any $\eta'' > 0$, and N large enough,

$$\frac{1}{Y_N} \geq e^{-N(\eta'' + \beta \inf_{\xi \in [b_-, b_+]} \mathcal{J}^{V;[b_-, b_+]}(\xi))}. \quad (1-4)$$

Putting Eqn. 1-3 and 1-4 together, we deduce that for all $\delta > 0$ and N large enough:

$$\mu_{N,\beta}^{V;[b_-,b_+]}(\lambda_{\min} \in F) \leq e^{N\beta(-\inf_{x \in F} \mathcal{J}^{V;[b_-,b_+]}(x) + \inf_{\xi \in [b_-, b_+]} \mathcal{J}^{V;[b_-,b_+]}(\xi) + \delta)},$$

which provides the announced upper bound.

Conclusion

As a consequence, since we assumed that the rate function only vanishes at α_-, α_+ we deduce that for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ so that:

$$\mu_{N,\beta}^{V;[b_-,b_+]}(\lambda_{\min} \leq \alpha_- - \epsilon) \leq e^{-\delta_\epsilon N}, \quad (1-5)$$

as well as a similar result for the largest eigenvalue.

A.4 Lower bound for large deviation of extreme eigenvalues

To establish a lower bound, we start again from Eqn. 4-12 with an open ball $B =]x - \epsilon, x + \epsilon[\subset [b_-, \alpha_-]$:

$$\mu_{N,\beta}^{V;[b_-,b_+]}(\lambda_{\min} \in B) = Y_N \int_B d\xi e^{-\frac{\beta}{2}V(\xi)} \Xi_N(\xi),$$

but replace the role of Y_N and Ξ_N in the bounds. Namely, we first have by Jensen's inequality:

$$\int_B d\xi e^{-\frac{\beta}{2}V(\xi)} \Xi_N(\xi) \geq \kappa_N e^{\int d\tilde{\chi}(\xi, \lambda) (\beta \sum_{i=1}^{N-1} \ln|\xi - \lambda_i| - \frac{\beta}{2}(N-1)V(\xi))},$$

with

$$\begin{aligned} d\tilde{\chi}(\xi, \lambda) &= \frac{\mathbf{1}_B(\xi) \mathbf{1}_{\lambda_{\min} \geq \xi}}{\kappa_N} d\xi e^{-\frac{\beta}{2}V(\xi)} d\mu_{N-1,\beta}^{V;[b_-,b_+]}(\lambda), \\ \kappa_N &= \int_B d\xi e^{-\frac{\beta}{2}V(\xi)} \mu_{N-1,\beta}^{V;[b_-,b_+]}[\mathbf{1}_{\lambda_{\min} \geq \xi}]. \end{aligned}$$

Thanks to Eqn. 1-5 (note that it applies similarly to $NV/(N-1)$ as the assumptions does not depend on the fine asymptotics of V), we know that κ_N converges towards a non vanishing constant. Moreover, the logarithm, once integrated against $d\xi$, produces a smooth bounded function and therefore we can use the convergence of L_{N-1} towards μ_{eq} under $\mu_{N-1,\beta}^{\frac{N}{N-1}V;[b_-,b_+]}$ to conclude that:

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int_B d\xi e^{-\frac{\beta}{2}V(\xi)} \Xi_N(\xi) \geq -\frac{\beta}{2} \frac{\int_B d\xi e^{-\frac{\beta}{2}V(\xi)} \left(V(\xi) - 2 \int d\mu_{\text{eq}}(\eta) \ln |\xi - \eta| \right)}{\int_B d\xi e^{-\frac{\beta}{2}V(\xi)}}.$$

Letting now ϵ going to zero in $B =]x - \epsilon, x + \epsilon[$ proves that:

$$\liminf_{\epsilon \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int_{B(x,\epsilon)} d\xi e^{-\frac{\beta}{2}V(\xi)} \Xi_N(\xi) \geq \beta \left(\int d\mu_{\text{eq}}(\eta) \ln |\xi - \eta| - \frac{V(\eta)}{2} \right). \quad (1-6)$$

To bound Y_N from below, it is enough to bound $1/Y_N$ from above, which can be done in the same way we bounded Ξ_N from above in the argument for the upper bound. We finally conclude:

$$\liminf_{\epsilon \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mu_{N,\beta}^{V;[b_-,b_+]} (\lambda_{\min} \in]x - \epsilon, x + \epsilon]) \geq -\beta \tilde{\mathcal{J}}^{V;[b_-,b_+]}(x),$$

which completes the proof of the large deviation principle.

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