18.440 Midterm 1 Solutions, Fall 2011: 50 minutes, 100 points

1. (20 points) Jill goes fishing. During each minute she fishes, there is a 1/600 chance that she catches a fish (independently of all other minutes). Assume that she fishes for 15 hours (900 minutes). Let N be the total number of fish she catches.

(a) Compute $E[N]$ and $\mathrm{Var}[N]$. (Give exact answers, not approximate ones.) **ANSWER:** By additivity of expectation $E[N] = 900/600 = 3/2$. By variance additivity for independent random variables $\mathrm{Var}[N] = 900(1/600)(599/600)$

(b) Compute the probability she catches exactly 3 fish. Give an *exact answer.* **ANSWER:** $\binom{900}{3}(1/600)^3(599/600)^{897}$

(c) Now use a Poisson random variable calculation to *approximate* the probability that she catches exactly 3 fish. **ANSWER:** N is approximately Poisson with $\lambda = 900/600 = 3/2$. So $P\{N = 3\} \approx e^{-\lambda}\lambda^3/3! = e^{-3/2}\frac{9}{16}$.

2. (10 points) Given ten people in a room, what is the probability that no two were born in the same month? (Assume that all of the 12^{10} ways of assigning birthday months to the ten people are equally likely.) **ANSWER:** $\frac{\binom{12}{10}10!}{12^{10}}$

3. (10 points) Suppose that X, Y and Z are independent random variables such that each is equal to 0 with probability .5 and 1 with probability .5.

(a) Compute the conditional probability $P[X + Y + Z = 1 | X - Y = 0]$. **ANSWER:** *Both* events occur if and only if both $X = Y = 0$ and $Z = 1$. So $P\{X + Y + Z = 1, X - Y = 0\} = 1/8$ and $P\{X - Y = 0\} = 1/2$. Thus $P[X + Y + Z = 1 | X - Y = 0] = (1/8)/(1/2) = 1/4$.

(b) Are the events $\{X = Y\}$ and $\{Y = Z\}$ and $\{X = Z\}$ independent? Are they pairwise independent? Explain. **ANSWER:** Not independent. Each event has probability 1/2 but probability all events occur is $1/4 \neq (1/2)^3$. Are pairwise independent, since probability of any two occurring is $(1/2)^2 = 1/4$.

4. (20 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability p.

(a) Let X be such that the first heads appears on the Xth toss. In other words, X is the number of tosses required to obtain a heads. Compute (in terms of p) the expectation and variance of X.
ANSWER: Recall or derive: $E[X] = \sum_{k=1}^{\infty} q^{k-1}pk$, where $q = 1 - p$. Cute trick: write $E[X-1] = \sum_{k=1}^{\infty} q^{k-1}p(k-1)$. Setting $j = k-1$, we have $E[X-1] = q\sum_{j=0}^{\infty} q^{j-1}pj = qE[X]$. Thus $E[X] - 1 = qE[X]$ and solving for $E[X]$ gives $E[X] = 1/(1-q) = 1/p$.

Similarly, recall or derive: $E[X^2] = \sum_{k=1}^{\infty} q^{k-1}pk^2$. Cute trick: $E[(X-1)^2] = \sum_{k=1}^{\infty} q^{k-1}p(k-1)^2$. Setting $j = k-1$, we have $E[(X-1)^2] = q\sum_{j=0}^{\infty} q^{j-1}pj^2 = qE[X^2]$. Thus $E[(X-1)^2] = E[X^2 - 2X + 1] = E[X^2] - 2/p + 1 = qE[X^2]$. Solving for $E[X^2]$ gives $(1-q)E[X^2] = pE[X^2] = 2/p - 1$, so $E[X^2] = (2-p)/p^2$ and $\mathrm{Var}[X] = \frac{1-p}{p^2}$.

(b) Let Y be such that the fifth heads appears on the Yth toss. Compute (in terms of p) the expectation and variance of Y. **ANSWER:** By additivity of expectation and variance (for independent random variables) we obtain $E[Y] = 5/p$ and $\mathrm{Var}[Y] = 5(1-p)/p^2$.

5. (20 points) Suppose that X is continuous random variable with probability density function $f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$. Compute the following:

(a) The expectation $E[X]$. **ANSWER:**
$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^{\infty} e^{-x}xdx = 1.$

(b) The probability $P\{X \in [-50, 50]\}$. **ANSWER:**
$P\{X \in [-50, 50]\} = \int_{-50}^{50} f_X(x)dx = \int_0^{50} e^{-x}dx = 1 - e^{-50}$

(c) The cumulative distribution function F_X. **ANSWER:**
$F_X(a) = \int_{-\infty}^{a} f_X(x)dx = \begin{cases} 0 & a \le 0 \\ \int_0^a e^{-x}dx = 1 - e^{-a} & a > 0 \end{cases}$

6. (20 points) A group of 52 people (labeled $1, 2, 3, \ldots, 52$) toss their hats into a box, mix them up, and return one hat to each person (all 52! permutations equally likely). Compute the following:

(a) The probability that the first 26 people all get their own hats.
ANSWER: $\frac{1}{52}\frac{1}{51}\cdots\frac{1}{27} = \frac{26!}{52!}$

(b) The probability that there are 26 pairs of people whose hats are switched: i.e., each pair can be labeled (a, b), such that a got b's hat and b got a's hat. **ANSWER:** Have $\binom{52}{2,2,2,\ldots,2} = 52!/(2^{26})$ ways to choose ordered list of 26 pairs. Dividing by 26! gives number of unordered collections of pairs. So we get $\frac{52!}{2^{26}26!}$ permutations of desired type. Dividing by 52! gives probability $\frac{1}{2^{26}26!}$.