Problem set 9, due December 5

The following exercises are taken from the book, references are from Ross, Edition 9, chapter 8 and 9.

Due monday december 1 at the beginning of lecture.

Problem 1 (Problem 6, Chap 8) A (fair) die is rolled until the total sum of all rolls exceeds 300. Approximate the probability that at least 80 rolls are necessary.

Problem 2 (Problem 7) Suppose that a fair die is rolled 100 times. Let X_i be the value of the ith roll. Compute an approximation for

$$\mathbb{P}\left(\prod_{i=1}^{100} X_i \leq a^{100}\right)$$

Problem 3 (Problem 9) If X is a gamma random variable with parameters $(n, 1)$, approximately how large need n be so that

$$\mathbb{P}\left(\left|\frac{1}{N}\sum_{i=1}^{n} X_i - 1\right| \geq 0.01\right) < .01?$$

Problem 4 (Problem 21extended) Prove that if X is a non-negative random variable

(1)
$$\mathbb{E}[X] \leq \mathbb{E}[X^2]^{1/2} \leq \mathbb{E}[X^3]^{1/3} \leq \cdots \leq \mathbb{E}[X^k]^{1/k} \leq \cdots$$

(2) For all $k \in \mathbb{N}$, all $a > 0$
$$\mathbb{P}\left(X \geq a\right) \leq \frac{\mathbb{E}[X^k]}{a^k}$$

(3) For $\lambda \geq 0$, prove that
$$\mathbb{P}(X \geq a) \leq \mathbb{E}[e^{\lambda X}]e^{-\lambda a}$$
Specialize to the case when X is an exponential variable with mean 1 and optimize the choice of λ.

(4) Let $(X_i, i \geq 0)$ be iid variables such that $\Lambda(t) = \mathbb{E}[e^{tX}]$ is finite for all t. Show that for all $a \in \mathbb{R}$
$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq a\right) \leq \inf_{t\geq 0}(\log \Lambda(t) - ta)$$

Problem 5 (Problem 1, Chap 9) Customers arrive at a bank following a Poisson Process with rate λ. Suppose that exactly 2 customers arrived during the first hour. What is the probability that

(1) Both arrived during the first 20 minutes ?
(2) At least one arrived during the first 20 minutes ?

Problem 6 (Problem 7, Chap 9) A transition probability matrix is said to be doubly stochastic if $\sum_{i=0}^{M} P_{ij} = 1$ for all states $j = 0, 1, \ldots, M$. Show that if such a Markov chain is ergodic, then the stationary measure π_j is uniform, that is $\pi_j = 1/(1 + M)$ for all j.

Problem 7 (Problem 8, Chap 9) On any given day, Buffy is either cheerful (c), so-so (s) or gloomy (g). If she is cheerful today, then she will be c,s or g tomorrow with respective probability 0.7,0.2 and 0.1. If she is (s) today, then she will be c,s or g with respective probability .4,.3 and .3. If she is gloomy today, then Buffy will be c, s, or g tomorrow with probability .2,.4 and .4. If Buffy is cheerful today, what is the probability that she will be cheerful in 3 days ? What proportion of time is Buffy cheerful ?