Problem set 8, due May 8

This homework is graded on 12 points; 1 point per exercise, except the two last which are 1.5 each. No collaboration for this homework. Do it on your own. Detail the arguments.

•(A) Below are 3 transition matrices. Draw the transition graphs, whether the chain is irreducible or not, and classify the states.

(1)

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/6 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

(3)

$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

•(B)**Bernoulli-Laplace diffusion model** Let two urns A and B filled with r red and r white marbles respectively. Consider a sequence of trials each consisting in drawing one marble from each urn and switching them. Let X_n be the number of red marbles in urn A after n trials, $n \in \mathbb{N}$. Find the transition matrix for the homogeneous Markov chain $\{X_n\}_{n\geq 1}$, and classify the states. Find the stationary distribution.

• (C)**Image of Markov chains** Let (X_k) be a homogeneous Markov process on a finite set S with transition matrix P. Fix a map $\Phi : S \to S'$ where S' is another finite set. Find a general sufficient condition on P and Φ such that $(\Phi(X_k))_{k\geq 0}$ is a Markov process.

•(D) **Transience versus recurrence** Let $f_1, f_2 \ldots,$ be a sequence of positive real numbers such that $\sum_{i\geq 1} f_i \leq 1$. Let $F_n = \sum_{i=1}^n f_i$, $F_0 = 0$, and the Markov chain with state space $\mathbb{Z}^+$ defined by the transition matrix $P = (P_{ij})$ with

$$P_{i0} = \frac{f_{i+1}}{1 - F_i} \qquad P_{ii+1} = 1 - P_{i0} = \frac{1 - F_{i+1}}{1 - F_i}$$

for $i \geq 0$ and $F_i < 1$. If $F_i = 1$, put $P_{ii+1} = 0$. Let q_l denote the probability that the Markov chain is in state 0 at time l and T_0 be the first return time to 0. Show that

1. $P(T_0 = l) = f_l$. Deduce a necessary and sufficient condition for transience. In the sequel assume that the chain is recurrent.
2. For $l \geq 1$ $q_l = \sum_k f_k q_{l-k}$
3. Find the necessary and sufficient conditions on the $f_i, i \geq 1$ for positive recurrence of the origin and the other integer numbers.
4. Describe a stationary solution for P in case of positive recurrence.

•(E) **Tennis game** Consider a game of tennis between two players A and B. Let us assume that A wins the points with probability p, and that points are won independent. In a game there is essentially 17 different states: 0-0, 15-0, 30-0, 40-0, 15-15, 30-15, 40-15, 0-15, 0-30, 0-40, 15-30, 15-40, advantage A, advantage B, game A, game B, deuce (30-30 and deuce, respectively 30-40 and advantage B, respectively 40-30 and advantage A may be considered to be the same state). Show that the probability for A winning the game, p_A, is

$$p_A = p^4 + 4p^4 q + \frac{10p^4q^2}{1-2pq} = \begin{cases} \frac{p^4(1-16q^4)}{p^4-q^4} & p \neq q \\ \frac{1}{2} \quad p = q. \end{cases}$$

where $q = 1 - p$.

• (F)**Heavy tail random walk** We consider a random walk on $\mathbb{Z}$ so that $S_n = \sum_{i=1}^n Y_i$ where Y_i are i.i.d random variables with values in $\mathbb{Z}$. We investigate recurrence when the Y_i may have infinite variance.

(1) Let $\phi(\theta) = \mathbb{E}[e^{i\theta Y_1}]$ for $\theta \in \mathbb{R}$. Show that

$$\mathbb{P}(S_n = 0) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta)^n d\theta\,.$$

(2) Show that

$$\sum_{n \geq 0} P(S_n = 0) = \lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} \Re(\frac{1}{1 - r\phi(\theta)}) d\theta$$

(3) Deduce a criterium for recurrence of the origin.

(4) Let Y be a discrete stable law, that is Y takes values in $\mathbb{N}$ and $\phi(\theta) = e^{-(1-e^{i\theta})^\alpha}$ for $\alpha \in (0, 1]$. Show that the random walk is transient.

(5) Assume that the above analysis extends to the case where the variables are real-valued and consider the case where the Y_i's are the α-stable variables so that $\phi(\theta) = e^{-|\theta|^\alpha}$, $\alpha \in (0, 2)$. Show that the random walk is recurrent iff $\alpha \geq 1$.

•(G)**Exercise 4.2.7 in Stroock book** Let $X_.$ be a Markov chain with transition probability P. Let i be a recurrent state and let

$$\mu_k = \mathbb{E}[\sum_{m=0}^{\rho_i - 1} 1_k(X_m) | X_0 = i] \in [0, \infty] \in [0, \infty]$$

(1) Show that $\mu P = \mu$.

(2) Show that if i is not positive recurrent, $\mu_i = 1$ and $\sum \mu_j = +\infty$ and if i is positive recurrent $\mu_j = 0$ unless j communicates with i, and that in the latter case $\mu_j \in (0, \infty)$.

(3) If i is positive recurrent, show that with C the communicating class of i

$$\bar{\mu} = \frac{\mu}{\sum \mu_k} = \pi^C$$

(H)**Poisson process**

(1) A Poisson process $N_t, t \geq 0$ with parameter λ is characterized by

$$N_t = \sum_{n \geq 1} 1_{\sum_{i=1}^n Z_i \leq t}$$

where Z_i are independent exponentially distributed variables with parameter λ.

Suppose buses arrive as a Poisson process of rate $\lambda > 0$ per hour. Suppose you arrive at the bus stop just as a bus pulls away. What is the expected time you have to wait until the next bus arrives? After waiting for an hour with no sign of the next bus, you start to get a bit fed up. What is the expected time you will have to wait now?

(2) Red cars pass a point on a road according to a Poisson process of rate λ and blue cars according to an independent Poisson process of rate μ. What is the probability that the first car that passes the point is red?

•(I)**Erdös-Rényi graph** We consider a random graph $\mathcal{G}_{n,p}$ with n vertices such that the number k of neighbors of a vertex $v \in \mathcal{G}_{n,p}$ is distributed as a binomial

$$B(n-1,p)(k) = \left(\begin{array}{c} n-1 \\ k \end{array} \right) p^k (1-p)^{n-1-k}$$

Equivalently, each edge is added randomly with probability p, independently of the other edges, and the vertices are labelled. We take $p = c/n$. We want to find the size of the largest connected component of this graph. To that end we consider the exploration process defined as follows: We first choose a vertex v. Then, we explore the neighbors of v ; they are then said to be explored (otherwise they are unexplored). In other words, the vertices have 3 states: unexplored (0), explored(1), saturated (2). Being given a configuration in $\{0,1,2\}^{|V|}$, our transition is deterministic: all neighbors of newly explored vertices become explored, all explored become saturated, the others stay the same. From each newly explored vertex, we explore all the vertices which have not yet been explored. The randomness comes from the underlying graph. Moreover, it should be understood that the exploration only explores from one vertex at a step. This vertex is decided based on the information of the already explored and saturated vertices in a deterministic way. We prove that if $c < 1$, the largest connected component has size $O(\log(n))$. To that end we prove that the exploration process stops after $O(\log(n))$ steps (a slightly more intricate argument using the same process could show that for $c > 1$ it is of size of order n)

(1) Show that if at time p there are $n-k$ unexplored vertices, the numbers of newly explored vertices at time $p+1$ follows $B(n-k, c/n)$.
(2) Show that if X_i are independent $B(n, c/n)$ variables,

$$\mathbb{P}(v \text{ belongs to a component of size } \geq k) \leq \mathbb{P}(\sum_{i=1}^{k} X_i \geq k-1)$$

(3) Deduce that

$$\lim_{n\to\infty} \mathbb{P}(\exists \text{ a component of size} \geq \frac{3}{(1-c)^2} \log n) = 0\,.$$

• (J)**Weighted deck of cards** We consider a deck of cards $\{1,\dots,n\}$ with probability $\omega_i > 0$ so that $\sum_{i=1}^n \omega_i = 1$. We denote the arrangements of the cards by a permutation of $\{1,\dots,n\}$: $\sigma(i)$ is the card at position i. Each time we pick the card i with probability w_i and put it at the top.

(1) Write down the transition probability matrix P on the space S_n of permutations of $\{1,\dots,n\}$.
(2) Show that it has a unique stationary measure π and describe it.
(3) Show by a coupling argument similar to that given in the course that

$$\|P^t(\sigma,.) - \pi\|_{TV} \leq \sum_{i=1}^n (1-w_i)^t\,.$$

•(K) **Coupling for the Glauber dynamics of the Potts model** The Potts model on the graph $G = (V,E)$ has state space $\Omega = \{1,\dots,q\}^{|V|}$ and is described by the measure

$$\pi_\beta(x) = \frac{1}{Z_\beta} \exp\{\beta \sum_{v\simeq w} 1_{x_v \neq w}\}$$

where $v \simeq w$ iff x and w are neighbors, β is a real number and Z_β the normalizing constant so that π_β is a probability measure. β is a real number. Consider the Glauber dynamics for π_β, that is the Markov chain so that at each time a vertex v is picked at random and its color in $c \in \{1, \ldots, q\}$ is then chosen randomly under the probability measure

$$P(x_v = c) = \frac{\exp\{\beta \sum_{w \simeq v} 1_{x_w \neq c}\}}{\sum_{q'} \exp\{\beta \sum_{w \simeq v} 1_{x_w \neq q'}\}}$$

Following the course, construct a Markovian coupling between X_n^x and X_n^y starting from x and y respectively, so that if β is small enough (in absolute value), there exists a constant $c < 1$ so that

$$\mathbb{E}[\rho(X_n^x, X_n^y)] \leq c^n \rho(x, y)$$

and deduce an upper bound on the mixing time of this chain. Here, ρ denotes the hamming distance $\rho(x, y) = \sum 1_{x_v \neq y_v}$.