# Problem set 6, due April 10

This homework is graded on 4 points; the 2 first exercises are graded on 0.5 point each, the 3 next on 1 point each.

• (A)**Coupling of two probability measures** Let $U, V$ be random variables on $\mathbb{N}$ with probability mass functions

$$f_U(x) = \frac{1}{2} 1_{\{0,1\}}(x), f_V(x) = \frac{1}{3} 1_{\{0,1,2\}}(x), x \in \mathbb{N}$$

where $1_S$ is the indicator function of the set $S$.

(1) Compute the total variation distance of $U$ and $V$.
(2) Give two different couplings of U and V .
(3) Give a coupling of $U$ and $V$ under which $U \leq V$ with probability 1.

•(B) **Markovian coupling** Let $(X_t, Y_t)$ be a Markovian coupling such that for some $0 < \alpha < 1$ and some $t_0 > 0$ the coupling time $\tau_{couple} = \min\{t > 0 : X_t = Y_t\}$ satisfies $P(\tau_{couple} \leq t_0) \geq \alpha$ for all pairs of initial states $(x, y)$. Prove that

$$E(\tau_{couple}) \leq \frac{t_0}{\alpha}$$

Hint: bound $P(kt_0 \leq \tau_{couple} \leq (k+1)t_0)$ by using the Markov property.

• (C) **Poisson coupling** Let $Y_i$ be independent Bernouilli with parameter $p_i$ ($P(Y_i = 1) = 1 - P(Y_i = 0) = p_i$), set $\lambda_i = -\log(1 - p_i)$ and $p_\lambda$ be a Poisson distribution

$$p_\lambda(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Let $P_n$ be the distribution of $X_n = \sum_{i=1}^n Y_i$. Then Show that if $\lambda = \sum_{i=1}^n \lambda_i$

$$\|P_n - p_\lambda\|_{TV} \leq \sum_{i=1}^n \lambda_i^2$$

*Hint:* Show that $Y_i$ has the same distribution as $Y_i' \wedge 1$ if $Y_i'$ is a Poisson variable with parameter $\lambda_i = -\log(1 - p_i)$ and that if the $Y_i'$ are independent $X_n' = \sum_{i=1}^n Y_i'$ is a Poisson variable with parameter $\lambda$. Use this remark to construct a coupling between $P_n$ and $p_\lambda$. Here, $\|.\|_{TV}$ is the total variation norm introduced in Levin, Peres, Wilmer book.

• (D) **The lazy random walk** Consider the lazy random walk on the $d$-dimensional torus $(\mathbb{Z} \backslash n\mathbb{Z})^d$ given for $i, j \in \mathbb{Z}^d$

$$P_{ij} = \frac{1}{2} \text{ if } i = j, \quad \frac{1}{4} \text{ if } i_k - j_k = r_k[n], \sum_{1 \leq k \leq d} |r_k| = 1$$

where $a = b[n]$ iff ($a - b$ is a multiple of $n$. We couple together a random walk $X_t$ started at $x$ with a random walk $Y_t$ started at $y$, first pick one of the d coordinates at random. If the positions of the two walks agree in the chosen coordinate, we move both of the walks by $+1, ?1$, or 0 in that coordinate, with probabilities $1/4, 1/4$ and $1/2$, respectively. If the positions of the two walks differ in the chosen coordinate, we randomly choose one of the chains to move (with probability 1/2) by $\pm 1$), leaving the other fixed.

Let $t \geq kdn^2$.

(1) Show that the probability that the first coordinate of the two walks have not yet coupled by time $t$ is less than $(1/4)^k$. *Hint:* Write down the evolution of the first coordinate and show that its coupling time has an expectation bounded by $dn^2/4$ uniformly and deduce that for the mixing time of the first coordinate

$$\bar{t}_{mix}(\frac{1}{4}) \leq dn^2 \, .$$

(2) By making an appropriate choice of $k$ and considering all the coordinates, obtain a bound of order $(dlogd)n^2$ on $t_{mix}(\frac{1}{4})$.

•(E)**Stationary time on a graph**

(1) Let $G = (V, E)$ be a graph and consider the lazy random walk on this graph (that is the Markov chain which stays put with probability $1/2$ and jump to a neighbor uniformly at each step). Let $W$ be the (random) vertex occupied at the first time the random walk has visited every vertex. That is, $W$ is the last new vertex to be visited by the random walk. Prove that for the random walk on the cycle, $W$ is uniformly distributed over all vertices different from the starting vertex. *Hint:* Either think that there should be a path going from a neighbor of $W$ to $W$ by going all along the cycle and argue that this does not depend on $W$ or compute the probability that a lazy random walk starting at 0 visits all the sites of $[-a, b]$ before exiting at $-a-1$ or $b+1$. For the later, show that the probability that the random walk starting at $i$ exits $[-a, b]$ in $b$ is $a + i/(a + b)$ by getting an induction relation. Conclude that the probability that the walk visits every vertex in $[a, b]$ before visiting vertex $-1-a$ or vertex $b+1$ is $1/a + b + 1$. Conclude.

(2) Consider again the lazy random walk on the cycle with length $n$. Define $\tau$ by tossing a coin with probability of heads $1/n$. If heads take $\tau = 0$ and otherwise take $\tau$ to be the first time every vertex has been visited. Show that $\tau$ is a stationary time. Show that it is not a strong stationary time.