Problem set 5, due thursday April 3

This homework is graded on 4 points; the first exercise is graded on 1 point, the second on 0,5 point, the third on 1 point, the fourth on 1.5 points. The final grade will be obtained by taking the minimum of 4 and the sum of the grades obtained in the 4 exercises.

•(A) **Formula for the stationary measure.** Let P be a transition probability on a finite state space $\mathbb{S}$. Show that

(1) $I-P$ has an eigenvalue equal to zero.

(2) Assume that $I-P$ has only one eigenvalue equal to zero. Show that P has a stationary measure and that it is given by

$$\pi_i = \frac{\det((I-P)^{\{i\}})}{\sum_j \det(I-P)^{\{j\}})}$$

where $M^{\{j\}}$ denotes the $(|\mathbb{S}|-1)\times((|\mathbb{S}|-1)$ obtained from M by removing the ith row and column.

(3) Let $P=((p_{i,j})_{1\le i,j\le 3})$ be a transition probability on $\{1,2,3\}$ and set

$$D(i,j,k) = p_{ji}(1-p_{kk}) + p_{jk}p_{ki}$$

Show that

(a)

$$\det((I-P)^{\{1\}}) = D(1,2,3), \quad \det((I-P)^{\{2\}}) = D(2,3,1),$$
$$\text{and } \det((I-P)^{\{3\}}) = D(3,1,2)\,.$$

(b) Conclude that P has a unique stationary distribution if and only if

$$\Pi := D(1,2,3) + D(2,3,1) + D(3,1,2) > 0$$

and then

$$\pi_1 = \frac{D(1,2,3)}{\Pi}, \quad \pi_2 = \frac{D(2,3,1)}{\Pi}, \quad \pi_3 = \frac{D(3,1,2)}{\Pi}\,.$$

•(B) **Balls in boxes.** Consider two boxes 1 and 2 containing a total of N balls. After the passage of each unit of time one ball is chosen randomly and moved to the other box. Consider the Markov chain with state space $\{0,1,2,,N\}$ representing the number of balls in box 1.

(1) What is the transition matrix of the Markov chain?

(2) Determine periodicity, transience, recurrence of the Markov chain.

• (C)**Constructing Markov chains with given stationary measure.** Let P be a transition probability on $\mathbb{S}$ finite. Let π be a probability vector with positive entries. In this exercise we construct two Markov chains with stationary measure π.

(1) *Metropolis chain.*

(a) Show that

$$\hat{P}_{i,j} = \begin{cases} P_{i,j}\min\{\frac{\pi_j}{\pi_i}\frac{P_{j,i}}{P_{i,j}},1\} & \text{if } j\neq i \\ 1-\sum_{k\neq i} P_{i,k}\min\{\frac{\pi_k}{\pi_i}\frac{P_{k,i}}{P_{i,k}},1\} & \end{cases}$$

is a transition probability matrix so that $\pi\hat{P}=\pi$.

(b) Assume $P_{i,j}=P_{j,i}$ is aperiodic and irreducible. Show that $\hat{P}^n_{i,j}$ converges towards π_j for any $j\in\mathbb{S}$.

(2) *Glauber dynamics.* Set $\mathbb{S} = S^V$ for two finite sets S, V and put for $v \in V$

$$\Omega(x,v) = \{y \in S^V : y_w = x_w \text{ for all} w \neq v\}$$

Put

$$\pi_{x,v}(y) = \begin{cases} \frac{\pi(y)}{\pi(\Omega(x,v))} \text{ if } y \in \Omega(x,v), \\ 0 \text{ if } y \notin \Omega(x,v) \end{cases}$$

Let $\tilde{P}$ be the Markov chain given by choosing independently at any step a vertex v with the uniform measure on V and choose a new configuration according to $\pi_{x,v}$.

(a) Describe the transition matrix $\tilde{P}$ and show that $\tilde{P}$ is a transition probability. Prove that π is stationary for $\tilde{P}$.

(b) Show that it satisfies the weak Doeblin condition and determine reccurence/transience of the Markov chain.

•(D)**Particles in a region** Consider a region D of space containing N particles. After the passage of each unit of time, each particle has probability $q \in (0,1)$ of leaving region D, and k new particles enter the region D following a Poisson distribution with parameter λ:

$$\mathbb{P}(x = k) = \frac{1}{k!}\lambda^k e^{-\lambda} \,.$$

The exit and entrance phenomena are assumed to be independent. Consider the Markov chain with state space $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ representing the number of particles in the region.

(1) Compute the transition matrix P for the Markov chain.

(2) Show that

$$\pi_k = e^{-\frac{\lambda}{q}} \frac{\lambda^k}{q^k k!} \,.$$

is stationary for P.

(3) Show that the Markov chain is irreducible.

(4) Show that there exists $B > 0$ such that $[0, B]$ is positive recurrent. *Hint:* Exhibit a Lyapounov function and use exercise D in Problem set 4.

(5) Show that for all j

$$\lim_{n\to\infty} P^n_{ij} = \pi_j$$

Hint: Show that $\|P(j,.) - P(i,.)\|_v \leq q|i-j|$ either by direct computation or by coupling techniques. Here $\|\mu\|_v = \frac{1}{2}\sum |\mu_i|$ is the total variation norm. *Hint*: Construct a coupling X_1^j, X_1^i of $P(j,.)$ and $P(i,.)$ so that

$$\mathbb{E}[|X_1^j - X_1^i|] \leq (1-q)|j-i| \,.$$