Problem set 2, due February 27

This homework is graded on 4 points; the 3 first exercises are graded on 0.5 point each, the third on 1 point and the last on 1.5 points.

• **The adjoint of a transition probability matrix.** Let P be a transition probability matrix with a stationary probability π. Assume $\pi_i > 0$ for all $i \in \mathbb{S}$. Define the *adjoint* of P with respect to π as the matrix P^T given for $i, j \in \mathbb{S}$ by

$$(P^T)_{ij} = \frac{\pi_j}{\pi_i} P_{ji}$$

(1) Show that P^T is a transition probability matrix for which π is also a stationary probability.
(2) Denote $\mathbb{P}$ and $\mathbb{P}^T$ to denote probabilities computed for the Markov chain with transition probability P and P^T respectively, with initial condition π. Show that these chains are the *reverse* of each others in the sense that for all $i_0, \ldots, i_n \in \mathbb{S}$

$$\mathbb{P}^T(X_0 = i_0, \ldots, X_n = i_n) = \mathbb{P}(X_n = i_0, \ldots, X_0 = i_n)$$

•**The stationary measure of a doubly stochastic matrix.** Let $\mathbb{S}$ be finite. Let P be a doubly stochastic matrix, that is a matrix with non-negative entries such that its rows and columns sum up to one. Under the condition that each entry is strictly positive, show that P has a unique equilibrium measure and describe it.

•**Simple random walk on $\mathbb{Z}$.** Let $X_n, n \geq 0$ be the random walk on $\mathbb{Z}$ with equal probability $1/2$ to go one step left or right. Show that $Y_n = |X_n|$ is a Markov chain and exhibit the transition matrix.

•**Coupling and total variation distance.** A coupling of two probability vectors μ, ν is a pair of random variables (X, Y) defined on a single probability space such that $\mathbb{P}(X = i) = \mu_i, \mathbb{P}(Y = j) = \nu_j$. Show that, with $\mu(A) = \sum_{i \in A} \mu_i$

(1)

$$\|\mu - \nu\|_{\mathrm{v}} = 2 \max_{A \subset \mathbb{S}} |\mu(A) - \nu(A)|$$

Hint: Show that if $B = \{x : \mu(x) \geq \nu(x)\}$ then for all $A \subset \mathbb{S}$, $\mu(A) - \nu(A) \leq \mu(B) - \nu(B)$.

(2) Show that

$$\|\mu - \nu\|_{\mathrm{v}} \leq 2 \inf\{\mathbb{P}(X \neq Y); (X, Y) \text{ coupling of } \mu \text{ and } \nu\}$$

Hint: Use the first point.

(3) Let ν (resp μ) be the probability vector on $\mathbb{S} = \mathbb{N}$ given as the law of $f(X)$ (resp. $g(\tilde{X})$) for $X, \tilde{X}$ standard Gaussian variables, that is

$$\mathbb{P}(X \geq t) = \mathbb{P}(\tilde{X} \geq t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-y^2/2} dy$$

and f, g functions from $\mathbb{R}$ into $\mathbb{N}$. In other words, $\mu_i = \int 1_{f(y)=i} e^{-y^2/2} dy / \sqrt{2\pi}$

Show that

$$\|\nu - \mu\|_{\mathrm{v}} \leq \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |f(y) - g(y)| e^{-y^2/2} dy \,.$$

•**Branching processes.** Take $\mathbb{S} = \mathbb{N}$. the state i represents the number of members in the population and the process evolves so that at each step every individual, independently of the others, dies and is replaced by a random number of offsprings distributed according to a probability vector $\mu = (\mu_1, \ldots, \mu_k, \ldots)$. Thus, 0 is an absorbing state (if the process reaches 0 it stays there for ever), and given that there are $i \geq 1$ individuals alive at a given time n, the number of individuals alive at time $n+1$ will be distributed as the sum of i mutually independent identically distributed random variables with law μ Let μ^{*m} so that $\mu^{*0}_j = \delta_{0,j}$ and for $m \geq 1$ set

$$\mu^{*m}_j = \sum_{i=0}^{j} \mu^{*(m-1)}_{j-i} \mu_i$$

and denote X_n the population at the nth step. Observe that μ^{*m} is the distribution of the number of individuals at time $n+1$ knowing that there are m individuals at time n for any $n \geq 0$, that is

$$\mathbb{P}(X_{n+1} = k | X_n = m) = \mu^{*m}_k \,.$$

Then the transition probability matrix of the branching process is Pgiven by $P_{ij} = (\mu^{*i})_j$. The first question is to predict extinction, that is compute $\lim_{n\to\infty} \mathbb{P}(X_n = 0)$. Assume that $\mu_0 > 0$ and $\mu_0 + \mu_1 < 1$ since otherwise the answer is trivial.

(1) Set $f(s) = \sum_{k=0}^{\infty} s^k \mu_k$ for $s \in [0,1]$ and define $f^{\circ n}$ so that $f^{\circ 0}(s) = s$ and $f^{\circ n} = f \circ f^{\circ(n-1)}$. Show that $\gamma = \sum_k k\mu_k = f'(1)$ and

$$\mathbb{E}[s^{X_n} | X_0 = i] = [f^{\circ n}(s)]^i = \sum_{j=0}^{\infty} s^j (P^n)_{ij} \,.$$

Hint: Begin by showing that $f(s)^i = \sum_j s^j (\mu^{*i})_j$.

(2) Observe that $s \to f(s) - s$ is continuous, positive at $s = 0$, null at $s = 1$, twice differentiable and strictly convex on $(0,1)$ (that is $f''(x) > 0$ for all $x \in [0,1]$). Conclude that either $\gamma \leq 1$ and $f(s) > s$ for all $s \in [0,1)$ or $\gamma > 1$ and there exists one $\alpha \in (0,1)$ at which $f(\alpha) = \alpha$.

(3) Deduce that

$$\gamma \leq 1 \Rightarrow \lim_{n\to\infty} \mathbb{E}[s^{X_n} | X_0 = i] = 1 \quad \text{ for all } s \in (0,1]$$

and

$$\gamma > 1 \Rightarrow \lim_{n\to\infty} \mathbb{E}[s^{X_n} | X_0 = i] = \alpha^i \quad \text{ for all } s \in (0,1)$$

(4) Conclude that $\gamma \leq 1$ implies

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0 | X_0 = i) = 1$$

whereas $\gamma > 1$ implies

$$\lim_{n\to\infty} P(X_n = 0 | X_0 = i) = \alpha^i$$

and

$$\lim_{n\to\infty} \mathbb{P}(1 \leq X_n \leq L | X_0 = i) = 0 \quad \text{ for all } L \geq 1$$

hence, when $\gamma > 1$ then either the population gets extinct or grows indefinitely.