3.5 Dimensions of the Four Subspaces

1 The column space $\boldsymbol{C}(A)$ and the row space $\boldsymbol{C}(A^{\mathrm{T}})$ both have *dimension* r (the rank of A).

2 The nullspace $\boldsymbol{N}(A)$ has *dimension* $n-r$. The left nullspace $\boldsymbol{N}(A^{\mathrm{T}})$ has *dimension* $m-r$.

3 Elimination produces bases for the row space and nullspace of A: They are the same as for R.

4 Elimination often changes the column space and left nullspace (but dimensions don't change).

5 **Rank one matrices**: $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ = column times row: $\boldsymbol{C}(A)$ has basis $\boldsymbol{u}$, $\boldsymbol{C}(A^{\mathrm{T}})$ has basis $\boldsymbol{v}$.

The main theorem in this chapter connects ***rank*** and ***dimension***. The ***rank*** of a matrix is the number of pivots. The ***dimension*** of a subspace is the number of vectors in a basis. We count pivots or we count basis vectors. *The rank of A reveals the dimensions of all four fundamental subspaces.* Here are the subspaces, including the new one.

Two subspaces come directly from A, and the other two from A^{T}:

Four Fundamental Subspaces

1. The ***row space*** is $\boldsymbol{C}(A^{\mathrm{T}})$, a subspace of $\mathbf{R}^n$.

2. The ***column space*** is $\boldsymbol{C}(A)$, a subspace of $\mathbf{R}^m$.

3. The ***nullspace*** is $\boldsymbol{N}(A)$, a subspace of $\mathbf{R}^n$.

4. The ***left nullspace*** is $\boldsymbol{N}(A^{\mathrm{T}})$, a subspace of $\mathbf{R}^m$. This is our new space.

In this book the column space and nullspace came first. We know $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A)$ pretty well. Now the other two subspaces come forward. The row space contains all combinations of the rows. *This row space of A is the column space of A^{T}.*

For the left nullspace we solve $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$—that system is n by m. *This is the nullspace of A^{T}.* The vectors $\boldsymbol{y}$ go on the *left* side of A when the equation is written $\boldsymbol{y}^{\mathrm{T}}A = \mathbf{0}^{\mathrm{T}}$. The matrices A and A^{T} are usually different. So are their column spaces and their nullspaces. But those spaces are connected in an absolutely beautiful way.

Part 1 of the Fundamental Theorem finds the dimensions of the four subspaces. One fact stands out: ***The row space and column space have the same dimension*** r. This number r is the **rank** of the matrix. The other important fact involves the two nullspaces:

$\boldsymbol{N}(A)$ ***and*** $\boldsymbol{N}(A^{\mathrm{T}})$ ***have dimensions*** $\boldsymbol{n-r}$ ***and*** $\boldsymbol{m-r}$***, to make up the full*** $\boldsymbol{n}$ ***and*** $\boldsymbol{m}$***.***

Part 2 of the Fundamental Theorem will describe how the four subspaces fit together (two in $\mathbf{R}^n$ and two in $\mathbf{R}^m$). That completes the "right way" to understand every $A\boldsymbol{x} = \boldsymbol{b}$. Stay with it—you are doing real mathematics.

The Four Subspaces for R

Suppose A is reduced to its row echelon form R. For that special form, the four subspaces are easy to identify. We will find a basis for each subspace and check its dimension. Then we watch how the subspaces change (two of them don't change!) as we look back at A. The main point is that *the four dimensions are the same for A and R.*

As a specific 3 by 5 example, look at the four subspaces for this echelon matrix R:

$$\begin{matrix} m=3 \\ n=5 \\ r=2 \end{matrix} \qquad \boldsymbol{R} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{5} & 0 & \mathbf{7} \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} \textbf{pivot rows } 1 \textbf{ and } 2 \\ \\ \textbf{pivot columns } 1 \textbf{ and } 4 \end{matrix}$$

The rank of this matrix is $r = 2$ (*two pivots*). Take the four subspaces in order.

1. The ***row space*** of R has dimension 2, matching the rank.

Reason: The first two rows are a basis. The row space contains combinations of all three rows, but the third row (the zero row) adds nothing new. So rows 1 and 2 span the row space $\boldsymbol{C}(R^{\mathrm{T}})$.

The pivot rows 1 and 2 are independent. That is obvious for this example, and it is always true. If we look only at the pivot columns, we see the r by r identity matrix. There is no way to combine its rows to give the zero row (except by the combination with all coefficients zero). So the r pivot rows are a basis for the row space.

The dimension of the row space is the rank r. The nonzero rows of R form a basis.

2. The ***column space*** of R also has dimension $r = 2$.

Reason: The pivot columns 1 and 4 form a basis for $\boldsymbol{C}(R)$. They are independent because they start with the r by r identity matrix. No combination of those pivot columns can give the zero column (except the combination with all coefficients zero). And they also span the column space. Every other (free) column is a combination of the pivot columns. Actually the combinations we need are the three special solutions!

Column 2 is 3 (column 1). The special solution is $(-3,1,0,0,0)$.

Column 3 is 5 (column 1). The special solution is $(-5,0,1,0,0,)$.

Column 5 is 7 (column 1) $+\,2$ (column 4). That solution is $(-7,0,0,-2,1)$.

The pivot columns are independent, and they span, so they are a basis for $\boldsymbol{C}(R)$.

The dimension of the column space is the rank r. The pivot columns form a basis.

3. The ***nullspace*** of R has dimension $n - r = 5 - 2$. There are $n - r = 3$ free variables. Here x_2, x_3, x_5 are free (no pivots in those columns). They yield the three special solutions to $R\boldsymbol{x} = \boldsymbol{0}$. Set a free variable to 1, and solve for x_1 and x_4.

$$\boldsymbol{s}_2 = \begin{bmatrix} \mathbf{-3} \\ \mathbf{1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{s}_3 = \begin{bmatrix} \mathbf{-5} \\ 0 \\ \mathbf{1} \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{s}_5 = \begin{bmatrix} \mathbf{-7} \\ 0 \\ 0 \\ \mathbf{-2} \\ \mathbf{1} \end{bmatrix}$$

$R\boldsymbol{x} = \boldsymbol{0}$ has the complete solution $\boldsymbol{x} = x_2\boldsymbol{s}_2 + x_3\boldsymbol{s}_3 + x_5\boldsymbol{s}_5$
The nullspace has dimension 3.

Reason: There is a special solution for each free variable. With n variables and r pivots, that leaves $n-r$ free variables and special solutions. The special solutions are independent, because they contain the identity matrix in rows 2, 3, 5. So $\boldsymbol{N}(R)$ has dimension $n - r$.

The nullspace has dimension $n - r$. The special solutions form a basis.

4. The ***nullspace of $\boldsymbol{R}^{\mathrm{T}}$*** (***left nullspace of $\boldsymbol{R}$***) has dimension $m - r = 3 - 2$.

Reason: The equation $R^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ looks for combinations of the columns of R^{T} (*the rows of* R) that produce zero. This equation $R^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ or $\boldsymbol{y}^{\mathrm{T}}R = \boldsymbol{0}^{\mathrm{T}}$ is

Left nullspace
Combination
of rows is zero

$$\begin{array}{r} y_1\,[1,\ 3,\ 5,\ 0,\ 7] \\ +y_2\,[0,\ 0,\ 0,\ 1,\ 2] \\ +y_3\,[0,\ 0,\ 0,\ 0,\ 0] \\ \hline [0,\ 0,\ 0,\ 0,\ 0] \end{array} \tag{1}$$

The solutions y_1, y_2, y_3 are pretty clear. We need $y_1 = 0$ and $y_2 = 0$. The variable y_3 is free (it can be anything). **The nullspace of $\boldsymbol{R}^{\mathrm{T}}$ contains all vectors $\boldsymbol{y}$** $= (0, 0, y_3)$.

In all cases R ends with $m - r$ zero rows. Every combination of these $m - r$ rows gives zero. These are the *only* combinations of the rows of R that give zero, because the pivot rows are linearly independent. So $\boldsymbol{y}$ in the left nullspace has $y_1 = 0, \ldots, y_r = 0$.

If A is m by n of rank r, its left nullspace has dimension $m - r$.

Why is this a "*left* nullspace"? The reason is that $R^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ can be transposed to $\boldsymbol{y}^{\mathrm{T}}R = \boldsymbol{0}^{\mathrm{T}}$. Now $\boldsymbol{y}^{\mathrm{T}}$ is a row vector to the *left* of R. You see the y's in equation (1) multiplying the rows. This subspace came fourth, and some linear algebra books omit it—but that misses the beauty of the whole subject.

In $\mathbf{R}^n$ the row space and nullspace have dimensions r and $n - r$ (adding to n).
In $\mathbf{R}^m$ the column space and left nullspace have dimensions r and $m - r$ (total m).

The Four Subspaces for A

We have a job still to do. ***The subspace dimensions for A are the same as for R.*** The job is to explain why. A is now any matrix that reduces to $R = \mathrm{rref}(A)$.

$$\textbf{This } A \textbf{ reduces to } R \qquad A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix} \qquad \text{Notice } \boldsymbol{C}(A) \neq \boldsymbol{C}(R)\,! \qquad (2)$$

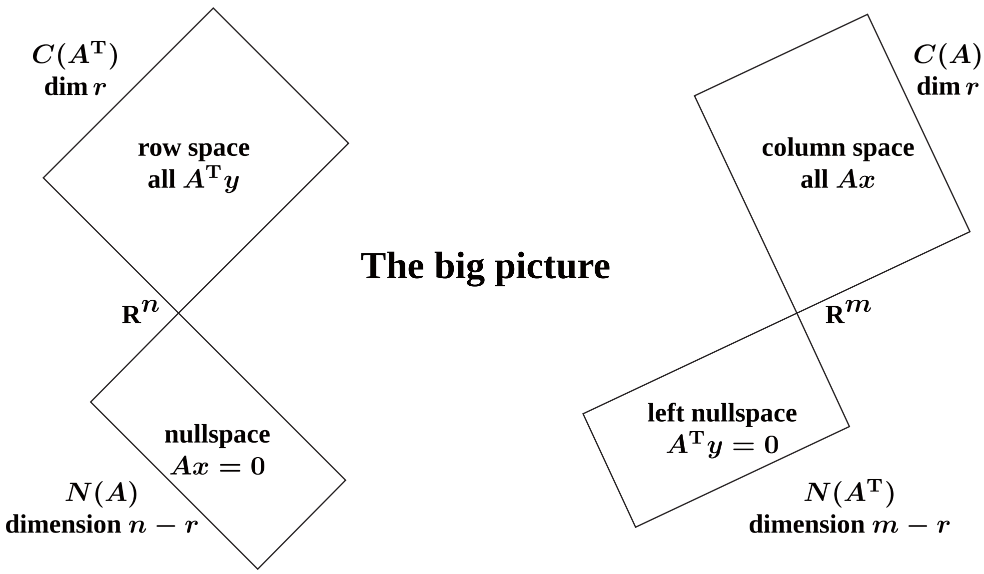


Figure 3.5: The dimensions of the Four Fundamental Subspaces (for R and for A).

1 ***A has the same row space as R.*** **Same dimension r and same basis.**

Reason: Every row of A is a combination of the rows of R. Also every row of R is a combination of the rows of A. Elimination changes rows, but not row *spaces*.

Since A has the same row space as R, we can choose the first r rows of R as a basis. Or we could choose r suitable rows of the original A. They might not always be the *first* r rows of A, because those could be dependent. The good r rows of A are the ones that end up as pivot rows in R.

2 ***The column space of A has dimension r. The column rank equals the row rank.***

Rank Theorem: The number of independent columns = the number of independent rows.

Wrong reason: "A and R have the same column space." This is false. *The columns of* R often end in zeros. The columns of A don't often end in zeros. Then $\boldsymbol{C}(A)$ is not $\boldsymbol{C}(R)$.

Right reason: The ***same*** combinations of the columns are zero (or nonzero) for A and R. Dependent in $A \Leftrightarrow$ dependent in R. Say that another way: $A\boldsymbol{x} = \mathbf{0}$ *exactly when* $R\boldsymbol{x} = \mathbf{0}$. The column spaces are different, but their *dimensions* are the same—equal to r.

Conclusion The r pivot columns of A are a basis for *its* column space $\boldsymbol{C}(A)$.

3 ***A has the same nullspace as R.*** **Same dimension $n - r$ and same basis.**

Reason: The elimination steps don't change the solutions. The special solutions are a basis for this nullspace (as we always knew). There are $n - r$ free variables, so the dimension of the nullspace is $n - r$. This is the **Counting Theorem**: $r + (n - r)$ equals n.

$$(\textbf{dimension of column space}) + (\textbf{dimension of nullspace}) = \textbf{dimension of } \mathbf{R}^{\boldsymbol{n}}.$$

4 ***The left nullspace of A*** (the nullspace of A^{T}) ***has dimension $m - r$.***

Reason: A^{T} is just as good a matrix as A. When we know the dimensions for every A, we also know them for A^{T}. Its column space was proved to have dimension r. Since A^{T} is n by m, the "whole space" is now $\mathbf{R}^m$. The counting rule for A was $r + (n - r) = n$. The counting rule for A^{T} is $r + (m - r) = m$. We now have all details of a big theorem:

***Fundamental Theorem of Linear Algebra*, Part 1**

The column space and row space both have dimension r.

The nullspaces have dimensions $n - r$ and $m - r$.

By concentrating on *spaces* of vectors, not on individual numbers or vectors, we get these clean rules. You will soon take them for granted—eventually they begin to look obvious. But if you write down an 11 by 17 matrix with 187 nonzero entries, I don't think most people would see why these facts are true:

Two key facts
dimension of $\boldsymbol{C}(A)$ = dimension of $\boldsymbol{C}(A^{\mathrm{T}})$ = rank of A
dimension of $\boldsymbol{C}(A)$ + dimension of $\boldsymbol{N}(A) = 17$.

Example 1 $A = [\,1 \quad 2 \quad 3\,]$ has $m = 1$ and $n = 3$ and rank $r = 1$.

The row space is a line in $\mathbf{R}^3$. The nullspace is the plane $A\boldsymbol{x} = x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 2 (which is $3 - 1$). The dimensions add to $\mathbf{1} + \mathbf{2} = \mathbf{3}$.

The columns of this 1 by 3 matrix are in $\mathbf{R}^1$! The column space is all of $\mathbf{R}^1$. The left nullspace contains only the zero vector. The only solution to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ is $\boldsymbol{y} = \mathbf{0}$, no other multiple of $[\,1 \quad 2 \quad 3\,]$ gives the zero row. Thus $\boldsymbol{N}(A^{\mathrm{T}})$ is $\mathbf{Z}$, the zero space with dimension 0 (which is $m - r$). In $\mathbf{R}^m$ the dimensions of $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ add to $\mathbf{1} + \mathbf{0} = \mathbf{1}$.

Example 2 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ has $m = 2$ with $n = 3$ and rank $r = 1$.

The row space is the same line through $(1, 2, 3)$. The nullspace must be the same plane $x_1 + 2x_2 + 3x_3 = 0$. The line and plane dimensions still add to $1 + 2 = 3$.

All columns are multiples of the first column $(1, 2)$. Twice the first row minus the second row is the zero row. Therefore $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has the solution $\boldsymbol{y} = (2, -1)$. The column space and left nullspace are **perpendicular lines** in $\mathbf{R}^2$. Dimensions $1 + 1 = 2$.

$$\text{Column space} = \text{line through} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{Left nullspace} = \text{line through} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

If A has three equal rows, its rank is _____. What are two of the $\boldsymbol{y}$'s in its left nullspace?

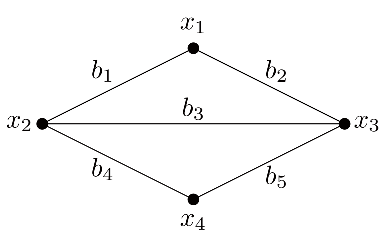
The y's in the left nullspace combine the rows to give the zero row.

Example 3 You have nearly finished three chapters with made-up equations, and this can't continue forever. Here is a better example of five equations (one for every edge in Figure 3.6). The five equations have four unknowns (one for every node). The matrix in $A\boldsymbol{x} = \boldsymbol{b}$ is an **incidence matrix**. This matrix A has 1 and -1 on every row.

Differences $A\boldsymbol{x} = \boldsymbol{b}$
across edges $1, 2, 3, 4, 5$
between nodes $1, 2, 3, 4$

$$\begin{array}{rrrrl} -x_1 & +x_2 & & & = b_1 \\ -x_1 & & +x_3 & & = b_2 \\ & -x_2 & +x_3 & & = b_3 \\ & -x_2 & & +x_4 & = b_4 \\ & & -x_3 & +x_4 & = b_5 \end{array} \qquad (3)$$

If you understand the four fundamental subspaces for this matrix (*the column spaces and the nullspaces for A and A^{T}*) you have captured the central ideas of linear algebra.



$$\boldsymbol{A} = \begin{bmatrix} \mathbf{-1} & \mathbf{1} & & \\ \mathbf{-1} & & \mathbf{1} & \\ & \mathbf{-1} & \mathbf{1} & \\ & \mathbf{-1} & & \mathbf{1} \\ & & \mathbf{-1} & \mathbf{1} \end{bmatrix} \quad \begin{array}{c} \text{edges} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

Figure 3.6: A "graph" with 5 edges and 4 nodes. A is its 5 by 4 incidence matrix.

The nullspace $\boldsymbol{N}(A)$ To find the nullspace we set $\boldsymbol{b} = \mathbf{0}$. Then the first equation says $x_1 = x_2$. The second equation is $x_3 = x_1$. Equation 4 is $x_2 = x_4$. *All four unknowns x_1, x_2, x_3, x_4 have the same value c.* The vectors $\boldsymbol{x} = (c, c, c, c)$ fill the nullspace of A.

That nullspace is a line in $\mathbf{R}^4$. The special solution $\boldsymbol{x} = (1, 1, 1, 1)$ is a basis for $\boldsymbol{N}(A)$. The dimension of $\boldsymbol{N}(A)$ is 1 (one vector in the basis). *The rank of A must be* 3, *since* $n - r = 4 - 3 = 1$. We now know the dimensions of all four subspaces.

The column space $C(A)$ There must be $r = 3$ independent columns. The fast way is to look at the first 3 columns. The systematic way is to find $R = \text{rref}(A)$.

$$\textbf{Columns 1, 2, 3 of } A \quad \begin{matrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{matrix} \qquad R = \textbf{reduced row echelon form} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From R we see again the special solution $\boldsymbol{x} = (1, 1, 1, 1)$. The first 3 columns are basic, the fourth column is free. To produce a basis for $\boldsymbol{C}(A)$ and not $\boldsymbol{C}(R)$, we go back to columns $1, 2, 3$ of A. The column space has dimension $r = 3$.

The row space $C(A^{\mathrm{T}})$ The dimension must again be $r = 3$. But the first 3 rows of A are *not independent*: row $3 =$ row $2 -$ row 1. So row 3 became zero in elimination, and row 3 was exchanged with row 4. *The first three independent rows are rows* $1, 2, 4$. Those three rows are a basis (one possible basis) for the row space.

I notice that edges $1, 2, 3$ form a **loop** in the picture: Dependent rows $1, 2, 3$. Edges $1, 2, 4$ form a **tree** in the picture. **Trees have no loops**! Independent rows $1, 2, 4$.

The left nullspace $N(A^{\mathrm{T}})$ Now we solve $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. Combinations of the rows give zero. We already noticed that row $3 =$ row $2 -$ row 1, so one solution is $\boldsymbol{y} = (1, -1, 1, 0, 0)$. I would say: That $\boldsymbol{y}$ comes from following the upper loop in the picture. Another $\boldsymbol{y}$ comes from going around the lower loop and it is $\boldsymbol{y} = (0, 0, \mathbf{-1}, \mathbf{1}, \mathbf{1})$: *row* $3 =$ *row* $4 +$ *row* 5. Those two $\boldsymbol{y}$'s are independent, they solve $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$, and the dimension of $\boldsymbol{N}(A^{\mathrm{T}})$ is $\boldsymbol{m} - \boldsymbol{r} = 5 - 3 = \mathbf{2}$. So we have a basis for the left nullspace.

You may ask how "loops" and "trees" got into this problem. That didn't have to happen. We could have used elimination to solve $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$. The 4 by 5 matrix A^{T} would have three pivot columns $1, 2, 4$ and two free columns $3, 5$. There are two special solutions and the nullspace of A^{T} has dimension two: $m - r = 5 - 3 = 2$. But *loops* and *trees* identify *dependent rows* and *independent rows* in a beautiful way. We use them in Section 10.1 for every incidence matrix like this A.

The equations $A\boldsymbol{x} = \boldsymbol{b}$ give "voltages" x_1, x_2, x_3, x_4 at the four nodes. The equations $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ give "currents" y_1, y_2, y_3, y_4, y_5 on the five edges. These two equations are **Kirchhoff's Voltage Law** and **Kirchhoff's Current Law**. Those words apply to an electrical network. But the ideas behind the words apply all over engineering and science and economics and business.

Graphs are *the most important model in discrete applied mathematics*. You see graphs everywhere: roads, pipelines, blood flow, the brain, the Web, the economy of a country or the world. We can understand their matrices A and A^{T}.

Rank One Matrices (Review)

Suppose every row is a multiple of the first row. Here is a typical example:

$$\begin{bmatrix} 2 & 3 & 7 & 8 \\ 2a & 3a & 7a & 8a \\ 2b & 3b & 7b & 8b \end{bmatrix} = \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} \begin{bmatrix} 2 & 3 & 7 & 8 \end{bmatrix} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$$

On the left is a matrix with three rows. But its row *space* only has dimension $= 1$. The row vector $\boldsymbol{v}^{\mathrm{T}} = \begin{bmatrix} 2 & 3 & 7 & 8 \end{bmatrix}$ tells us a basis for that row space. *The row rank is* 1.

Now look at the columns. "The column rank equals the row rank which is 1." All columns of the matrix must be multiples of one column. Do you see that this key rule of linear algebra is true? The column vector $\boldsymbol{u} = (1, a, b)$ is multiplied by $2, 3, 7, 8$. That nonzero vector $\boldsymbol{u}$ is a basis for the column space. *The column rank is also* 1.

Every rank one matrix is one column times one row $\quad \boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^{\mathbf{T}}$

Rank Two Matrices = Rank One plus Rank One

Here is a matrix A of rank $r = 2$. We can't see r immediately from A. So we reduce the matrix by row operations to $R = \mathrm{rref}(A)$. Some elimination matrix E simplifies A to $\boldsymbol{EA} = \boldsymbol{R}$. Then the inverse matrix $C = E^{-1}$ connects R back to $\boldsymbol{A} = \boldsymbol{CR}$.

You know the main point already: $\boldsymbol{R}$ **has the same row space as** $\boldsymbol{A}$.

Rank two

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 7 \\ 4 & 2 & 20 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{4} & \mathbf{2} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{4} \\ 0 & 0 & 0 \end{bmatrix} = CR. \qquad (4)$$

The row space of R clearly has two basis vectors $\boldsymbol{v}_1^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$ and $\boldsymbol{v}_2^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 4 \end{bmatrix}$. So the (same!) row space of A also has this basis: *row rank* $= 2$. Multiplying C times R says that row 3 of A is $4\boldsymbol{v}_1^{\mathrm{T}} + 2\boldsymbol{v}_2^{\mathrm{T}}$.

Now look at columns. The pivot columns of R are clearly $(1, 0, 0)$ and $(0, 1, 0)$. Then the pivot columns of A are also in columns 1 and 2: $\boldsymbol{u}_1 = (1, 1, 4)$ and $\boldsymbol{u}_2 = (0, 1, 2)$. Notice that C has those same first two columns! That was guaranteed since multiplying by two columns of the identity matrix (in R) won't change the pivot columns $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

When you put in letters for the columns and rows, you see **rank 2 = rank 1 + rank 1**.

Matrix $\boldsymbol{A}$ Rank two

$$A = \begin{bmatrix} & & \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \\ & & \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \\ \textbf{zero row} \end{bmatrix} = \boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}} = (\text{rank } 1) + (\text{rank } 1).$$

Did you see that last step? I multiplied the matrices using **columns times rows**. That was perfect for this problem. *Every rank r matrix is a sum of r rank one matrices*: Pivot columns of A times nonzero rows of R. The row $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ simply disappeared.

The pivot columns $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are a basis for the column space, which you knew.

■ REVIEW OF THE KEY IDEAS ■

1. The r pivot rows of R are a basis for the row spaces of R and A (same space).
2. The r pivot columns of A (!) are a basis for its column space $\boldsymbol{C}(A)$.
3. The $n-r$ special solutions are a basis for the nullspaces of A and R (same space).
4. If $EA=R$, the last $m-r$ rows of E are a basis for the left nullspace of A.

Note about the four subspaces The Fundamental Theorem looks like pure algebra, but it has very important applications. My favorites are the networks in Chapter 10 (often I go to 10.1 for my next lecture). The equation for $\boldsymbol{y}$ in the left nullspace is $A^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$:

Flow into a node equals flow out. Kirchhoff's Current Law is the "balance equation".

This must be the most important equation in applied mathematics. All models in science and engineering and economics involve a balance—of force or heat flow or charge or momentum or money. That balance equation, plus Hooke's Law or Ohm's Law or some law connecting "potentials" to "flows", gives a clear framework for applied mathematics.

My textbook on *Computational Science and Engineering* develops that framework, together with algorithms to solve the equations: Finite differences, finite elements, spectral methods, iterative methods, and multigrid.

■ WORKED EXAMPLES ■

3.5 A Put four 1's into a 5 by 6 matrix of zeros, keeping the dimension of its *row space* as small as possible. Describe all the ways to make the dimension of its *column space* as small as possible. Describe all the ways to make the dimension of its *nullspace* as small as possible. How to make the *sum of the dimensions of all four subspaces small*?

Solution The rank is 1 if the four 1's go into the same row, or into the same column. They can also go into *two rows and two columns* (so $a_{ii}=a_{ij}=a_{ji}=a_{jj}=1$). Since the column space and row space always have the same dimensions, this answers the first two questions: Dimension 1.

The nullspace has its smallest possible dimension $6-4=2$ when the rank is $r=4$. To achieve rank 4, the 1's must go into four different rows and four different columns.

You can't do anything about the sum $r+(n-r)+r+(m-r)=\boldsymbol{n}+\boldsymbol{m}$. It will be $6+5=11$ no matter how the 1's are placed. The sum is 11 even if there aren't any 1's...

If all the other entries of A are 2's instead of 0's, how do these answers change?

3.5 B Fact: All the rows of AB are combinations of the rows of B. So the row space of AB is contained in (possibly equal to) the row space of B. **Rank** $(\boldsymbol{AB}) \leq$ **rank** $(\boldsymbol{B})$.

All columns of AB are combinations of the columns of A. So the column space of AB is contained in (possibly equal to) the column space of A. **Rank** $(\boldsymbol{AB}) \leq$ **rank** $(\boldsymbol{A})$.

If we multiply by an *invertible* matrix, the rank will not change. The rank can't drop, because when we multiply by the inverse matrix the rank can't jump back.

Problem Set 3.5

1 (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?

2 Find bases and dimensions for the four subspaces associated with A and B:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

3 Find a basis for each of the four subspaces associated with A:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4 Construct a matrix with the required property or explain why this is impossible:

(a) Column space contains $\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$, row space contains $\begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix}, \begin{bmatrix} \mathbf{2} \\ \mathbf{5} \end{bmatrix}$.

(b) Column space has basis $\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{3} \end{bmatrix}$, nullspace has basis $\begin{bmatrix} \mathbf{3} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}$.

(c) Dimension of nullspace $= 1 +$ dimension of left nullspace.

(d) Nullspace contains $\begin{bmatrix} \mathbf{1} \\ \mathbf{3} \end{bmatrix}$, column space contains $\begin{bmatrix} \mathbf{3} \\ \mathbf{1} \end{bmatrix}$.

(e) Row space = column space, nullspace $\neq$ left nullspace.

5 If $\mathbf{V}$ is the subspace spanned by $(1,1,1)$ and $(2,1,0)$, find a matrix A that has $\mathbf{V}$ as its row space. Find a matrix B that has $\mathbf{V}$ as its nullspace. Multiply AB.

6 Without using elimination, find dimensions and bases for the four subspaces for

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$$

7 Suppose the 3 by 3 matrix A is invertible. Write down bases for the four subspaces for A, and also for the 3 by 6 matrix $B = [\, A \quad A \,]$. (The basis for $\boldsymbol{Z}$ is empty.)

8 What are the dimensions of the four subspaces for A, B, and C, if I is the 3 by 3 identity matrix and 0 is the 3 by 2 zero matrix?

$$A = \begin{bmatrix} I & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0^{\mathrm{T}} & 0^{\mathrm{T}} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 \end{bmatrix}.$$

9 Which subspaces are the same for these matrices of different sizes?

(a) $\begin{bmatrix} A \end{bmatrix}$ and $\begin{bmatrix} A \\ A \end{bmatrix}$ (b) $\begin{bmatrix} A \\ A \end{bmatrix}$ and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$.

Prove that all three of those matrices have the *same rank* r.

10 If the entries of a 3 by 3 matrix are chosen randomly between 0 and 1, what are the most likely dimensions of the four subspaces? What if the random matrix is 3 by 5?

11 (Important) A is an m by n matrix of rank r. Suppose there are right sides $\boldsymbol{b}$ for which $A\boldsymbol{x} = \boldsymbol{b}$ has *no solution*.

(a) What are all inequalities ($<$ or $\leq$) that must be true between m, n, and r?

(b) How do you know that $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has solutions other than $\boldsymbol{y} = \mathbf{0}$?

12 Construct a matrix with $(1, 0, 1)$ and $(1, 2, 0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

13 True or false (with a reason or a counterexample):

(a) If $m = n$ then the row space of A equals the column space.

(b) The matrices A and $-A$ share the same four subspaces.

(c) If A and B share the same four subspaces then A is a multiple of B.

14 Without computing A, find bases for its four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

15 If you exchange the first two rows of A, which of the four subspaces stay the same? If $\boldsymbol{v} = (1, 2, 3, 4)$ is in the left nullspace of A, write down a vector in the left nullspace of the new matrix after the row exchange.

16 *Explain why* $\boldsymbol{v} = (1, 0, -1)$ *cannot be a row of* A *and also in the nullspace*.

17 Describe the four subspaces of $\mathbf{R}^3$ associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

18 (Left nullspace) Add the extra column $\boldsymbol{b}$ and reduce A to echelon form:

$$\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right side.) Which vectors are in the nullspace of A^{T} and which vectors are in the nullspace of A?

19 Following the method of Problem 18, reduce A to echelon form and look at zero rows. The $\boldsymbol{b}$ column tells which combinations you have taken of the rows:

(a) $\begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$

From the $\boldsymbol{b}$ column after elimination, read off $m-r$ basis vectors in the left nullspace. Those $\boldsymbol{y}$'s are combinations of rows that give zero rows in the echelon form.

20 (a) Check that the solutions to $A\boldsymbol{x} = \mathbf{0}$ are perpendicular to the rows of A:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = ER.$$

(b) How many independent solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$? Why does $\boldsymbol{y}^{\mathrm{T}} =$ row 3 of E^{-1}?

21 Suppose A is the sum of two matrices of rank one: $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$.

(a) Which vectors span the column space of A?

(b) Which vectors span the row space of A?

(c) The rank is less than 2 if _____ or if _____.

(d) Compute A and its rank if $\boldsymbol{u} = \boldsymbol{z} = (1, 0, 0)$ and $\boldsymbol{v} = \boldsymbol{w} = (0, 0, 1)$.

22 Construct $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ whose column space has basis $(1, 2, 4), (2, 2, 1)$ and whose row space has basis $(1, 0), (1, 1)$. Write A as (3 by 2) times (2 by 2).

23 Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A cannot be invertible?

24 (Important) $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ is solvable when $\boldsymbol{d}$ is in which of the four subspaces? The solution $\boldsymbol{y}$ is unique when the _____ contains only the zero vector.

25 True or false (with a reason or a counterexample):

(a) A and A^{T} have the same number of pivots.

(b) A and A^{T} have the same left nullspace.

(c) If the row space equals the column space then $A^{\mathrm{T}} = A$.

(d) If $A^{\mathrm{T}} = -A$ then the row space of A equals the column space.

26 If a, b, c are given with $a \neq 0$, how would you choose d so that $\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$ has rank 1? Find a basis for the row space and nullspace. Show they are perpendicular!

27 Find the ranks of the 8 by 8 checkerboard matrix B and the chess matrix C:

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ & & & \text{four zero rows} & & & & \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

The numbers r, n, b, q, k, p are all different. Find bases for the row space and left nullspace of B and C. Challenge problem: Find a basis for the nullspace of C.

28 Can tic-tac-toe be completed (5 ones and 4 zeros in A) so that rank $(A) = 2$ but neither side passed up a winning move?

Challenge Problems

29 If $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ is a 2 by 2 matrix of rank 1, redraw Figure 3.5 to show clearly the Four Fundamental Subspaces. If B produces those same four subspaces, what is the exact relation of B to A?

30 $\mathbf{M}$ is the space of 3 by 3 matrices. Multiply every matrix X in $\mathbf{M}$ by

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \quad \text{Notice: } A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Which matrices X lead to $AX =$ zero matrix?

(b) Which matrices have the form AX for some matrix X?

(a) finds the "nullspace" of that operation AX and (b) finds the "column space". What are the dimensions of those two subspaces of $\mathbf{M}$? Why do the dimensions add to $(n - r) + r = 9$?

31 Suppose the m by n matrices A and B have *the same four subspaces*. If they are both in row reduced echelon form, prove that F must equal G:

$$A = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} I & G \\ 0 & 0 \end{bmatrix}.$$