Matrix Factorizations

1. \( A = CR \) (basis for column space of \( A \) (basis for row space of \( A \))

   **Requirements:** \( C \) is \( m \) by \( r \) and \( R \) is \( r \) by \( n \). Columns of \( A \) go into \( C \) if they are not combinations of earlier columns of \( A \). \( R \) contains the nonzero rows of the reduced row echelon form \( R_0 = \text{rref}(A) \). Those rows begin with an \( r \) by \( r \) identity matrix, so \( R \) equals \([ I \ F] \) times a column permutation \( P \).

2. \( A = CRM^* \)

   **Requirements:** \( C \) and \( R^* \) come directly from \( A \). Those columns and rows meet in the \( r \) by \( r \) matrix \( W = M^{-1} \) (Section 3.2): \( M = \text{mixing matrix} \). The first \( r \) by \( r \) invertible submatrix \( W \) is the intersection of the \( r \) columns of \( C \) with the \( r \) rows of \( R^* \).

3. \( A = LU \)

   **Requirements:** No row exchanges as Gaussian elimination reduces square \( A \) to \( U \).

4. \( A = LDU \)

   **Requirements:** No row exchanges. The pivots in \( D \) are divided out from rows of \( U \) to leave \( 1 \)'s on the diagonal of \( U \). If \( A \) is symmetric then \( U = L^T \) and \( A = LDL^T \).

5. \( PA = LU \) (permutation matrix \( P \) to avoid zeros in the pivot positions).

   **Requirements:** \( A \) is invertible. Then \( P, L, U \) are invertible. \( P \) does all of the row exchanges on \( A \) in advance, to allow normal \( LU \). Alternative: \( A = L_1 P_1 U_1 \).

6. \( S = CTC = (\text{lower triangular}) (\text{upper triangular}) \text{ with } \sqrt{D} \text{ on both diagonals} \)

   **Requirements:** \( S \) is symmetric and positive definite (all \( n \) pivots in \( D \) are positive). This *Cholesky factorization* \( C = \text{chol}(S) \) has \( C^T = L \sqrt{D} \), so \( S = C^T C = LDL^T \).

7. \( A = QR \) (orthonormal columns in \( Q \) (upper triangular matrix \( R \)).

   **Requirements:** \( A \) has independent columns. Those are *orthogonalized* in \( Q \) by the Gram-Schmidt or Householder process. If \( A \) is square then \( Q^{-1} = Q^T \).

8. \( A = X\Lambda X^{-1} \) (eigenvectors in \( X \)) (eigenvalues in \( \Lambda \)) (left eigenvectors in \( X^{-1} \)).

   **Requirements:** \( A \) must have \( n \) linearly independent eigenvectors.

9. \( S = QAQ^T \) (orthogonal matrix \( Q \)) (real eigenvalue matrix \( \Lambda \)) (\( Q^T \) is \( Q^{-1} \)).

   **Requirements:** \( S \) is *real and symmetric*: \( S^T = S \). This is the Spectral Theorem.
10. \( A = BJB^{-1} \) (generalized eigenvectors in \( B \)) (Jordan blocks in \( J \) \( (B^{-1}) \)).

**Requirements:** \( A \) is any square matrix. This *Jordan form* \( J \) has a block for each linearly independent eigenvector of \( A \). Every block has only one eigenvalue.

11. \( A = UΣV^{T} = \begin{pmatrix} \text{orthogonal} & \left( \begin{array}{c} m \times n \text{ singular value matrix} \\ \sigma_{1}, \ldots, \sigma_{r} \text{ on its diagonal} \end{array} \right) & \text{orthogonal} \end{pmatrix} \)

\( U \) is \( m \times m \)

\( V \) is \( n \times n \).

**Requirements:** None. This *Singular Value Decomposition* (SVD) has the eigenvectors of \( AA^{T} \) in \( U \) and eigenvectors of \( A^{T}A \) in \( V \); \( \sigma_{i} = \sqrt{\lambda_{i}(A^{T}A)} = \sqrt{\lambda_{i}(AA^{T})} \).

Those singular values are \( \sigma_{1} \geq \sigma_{2} \geq \cdots \geq \sigma_{r} > 0 \). By column-row multiplication

\[ A = UΣV^{T} = \sigma_{1}u_{1}v_{1}^{T} + \cdots + \sigma_{r}u_{r}v_{r}^{T}. \]

If \( S \) is symmetric positive definite then \( U = V = Q \) and \( \Sigma = \Lambda \) and \( S = QΛQ^{T} \).

12. \( A^{+} = VΣ^{+}U^{T} = \begin{pmatrix} \text{orthogonal} & \left( \begin{array}{c} n \times m \text{ pseudoinverse of } \Sigma \\ 1/\sigma_{1}, \ldots, 1/\sigma_{r} \text{ on diagonal} \end{array} \right) & \text{orthogonal} \end{pmatrix} \)

\( n \times n \)

\( m \times m \).

**Requirements:** None. The *pseudoinverse* \( A^{+} \) has \( A^{+}A = \) projection onto row space of \( A \) and \( AA^{+} = \) projection onto column space. \( A^{+} = A^{-1} \) if \( A \) is invertible. The shortest least-squares solution to \( Ax = b \) is \( x^{+} = A^{+}b \). This solves \( A^{T}Ax^{+} = A^{T}b \).

13. \( A = QS = \) (orthogonal matrix \( Q \)) (symmetric positive definite matrix \( S \)).

**Requirements:** \( A \) is invertible. This *polar decomposition* has \( S^{2} = A^{T}A \). The factor \( S \) is semidefinite if \( A \) is singular. The reverse polar decomposition \( A = KQ \) has \( K^{2} = AA^{T} \). Both have \( Q = UV^{T} \) from the SVD.

14. \( A = UΛU^{-1} = \) (unitary \( U \)) (eigenvalue matrix \( Λ \)) \( (U^{-1} \text{ which is } U^{H} = U^{T}) \).

**Requirements:** \( A \) is normal: \( A^{H}A = AA^{H} \). Its orthonormal (and possibly complex) eigenvectors are the columns of \( U \). Complex \( λ \)'s unless \( S = S^{H} \): Hermitian case.

15. \( A = QTQ^{-1} = \) (unitary \( Q \)) (triangular \( T \) with \( λ \)'s on diagonal) \( (Q^{-1} = Q^{H}) \).

**Requirements:** *Schur triangularization* of any square \( A \). There is a matrix \( Q \) with orthonormal columns that makes \( Q^{-1}AQ \) triangular: Section 6.3.

16. \( F_{n} = \left[ \begin{array}{cc} I & D \\ I & -D \end{array} \right] \left[ \begin{array}{c} F_{n/2} \\ F_{n/2} \end{array} \right] \left[ \begin{array}{c} \text{even-odd} \\ \text{permutation} \end{array} \right] = \) one step of the recursive FFT.

**Requirements:** \( F_{n} = \) Fourier matrix with entries \( w^{jk} \) where \( w^{n} = 1 \): \( F_{n}F_{n}^{H} = nI \).

\( D \) has \( 1, w, \ldots, w^{n/2 - 1} \) on its diagonal. For \( n = 2^{\ell} \) the *Fast Fourier Transform* will compute \( F_{n}x \) with only \( \frac{1}{2}n\ell = \frac{1}{2}n\log_{2}n \) multiplications from \( \ell \) stages of \( D \)'s.