# INTRODUCTION

# TO

# LINEAR

# ALGEBRA

# Fifth Edition

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MANUAL FOR INSTRUCTORS

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## Problem Set 12.1, page 544

**1** When 7 is added to every output, the mean increases by 7 and the variance does not change (because new variance comes from (distance)$^2$ to the new mean).

New sample mean and new expected mean : Add 7. New variance : No change.

**2** If we add $\frac{1}{3}$ to $\frac{1}{7}$ (fraction of integers divisible by 3 ***plus*** fraction divisible by 7) we have **double counted** the integers divisible by both 3 and 7. This is a fraction $\frac{1}{21}$ of all integers (because these double counted numbers are multiples of 21). So the fraction divisible by 3 or 7 or both is

$$\frac{1}{3}+\frac{1}{7}-\frac{1}{21}=\frac{7}{21}+\frac{3}{21}-\frac{1}{21}=\frac{9}{21}=\boldsymbol{\frac{3}{7}}.$$

**3** In the numbers from 1 to 1000, each group of ten numbers will contain each possible ending $x = 1, 2, 3, \ldots, 0$. So those endings all have the same probability $p_i = \frac{1}{10}$. Expected mean of that last digit $x$ :

$$m = \mathrm{E}\,[x] = \sum p_i\, x_i = \frac{1}{10}\sum_{i=0}^{9} i = \frac{45}{10} = \mathbf{4.5}$$

The best way to find the variance $\sigma^2 = 8.25$ is **in the last line below and in problem 12.1.7**. The slower way to find $\sigma^2$ is

$$\sigma^2 = \mathrm{E}\,[(x-4.5)^2] = \sum_{i=0}^{9} p_i(x_i-4.5)^2 = \frac{1}{10}\sum_{i=0}^{9}(i-4.5)^2$$

We can separate $\left(i-4.5\right)^2$ into $\left(i^2-9i+(4.5)^2\right)$ and add from $i=0$ to $i=9$ :

$$\frac{1}{10}\left(\sum_{0}^{9} i^2 - 9\sum_{0}^{9} i + \sum_{0}^{9}(4.5)^2\right) = \frac{1}{10}\left(285-9(45)+10(4.5)^2\right)$$
$$= \frac{1}{10}(285-405+202.5) = \frac{82.5}{10} = 8.25 = \frac{33}{4}.$$

Notice that 202.5 is half of 405—like $Nm^2$ and $2Nm^2$ in equation (4), page 536. **I should have extended equation** (4) **to its best form :**

$$\boldsymbol{\sigma^2 = \mathrm{E}\,[(x-m)^2] = \mathrm{E}\,[x^2] - m^2}$$

That quickly gives $\frac{285}{10} - (4.5)^2 = 8.25 = \text{same answer}$.

**4** For numbers ending in $0, 1, 2, \ldots, 9$ the squares end in $x = 0, 1, 4, 9, 6, 5, 6, 9, 4, 1$. So the probabilities of $x = 0$ and $5$ are $p = \frac{1}{10}$ and the probabilities of $x = 1, 4, 6, 9$ are $p = \frac{1}{5}$. The mean is

$$m = \sum p_i\, x_i = \frac{0}{0} + \frac{5}{10} + \frac{1}{5}\,(1 + 4 + 6 + 9) = 4.5 = \text{ same as before.}$$

The variance using the improvement of equation $(4)$ is

$$\begin{aligned} \sigma^2 &= \mathrm{E}\,[x^2] - m^2 = \frac{1}{10}\,0^2 + \frac{1}{10}\,5^2 + \frac{1}{5}\,(1^2 + 4^2 + 6^2 + 9^2) - m^2 \\ &= \frac{25}{10} + \frac{134}{5} - 20.25 = \mathbf{9.05} \end{aligned}$$

**5** For numbers from $1$ to $1000$, the first digit is $x = 1$ for $1000$ and 100-199 and 10-19 and $1$ (112 times). The first digit is $x = 2$ for 200-299 and 20-29 and $2$ (111 times). The other first digits $x = 3$ to $9$ also happen (111 times). Total (1000 times) is correct.

The average first digit is the mean, close to $\frac{1}{9}\big(1 + 2 + \cdots + 9\big) = 5$:

$$m = \sum p_i\, x_i = \frac{112}{1000}\,(1) + \frac{111}{1000}\,(2+3+\cdots+9) = \frac{112 + 111(44)}{1000} = \frac{4996}{1000} = 4.996 \approx 5.$$

The variance is

$$\sigma^2 = \mathrm{E}\,[(x - m)^2] = \mathrm{E}\,[x^2] - m^2 = \frac{112}{1000}\,(1^2) + \frac{111}{1000}(2^2 + \cdots + 9^2) - m^2$$

$$= \frac{112 + 111(284)}{1000} - m^2 \approx \frac{31635}{1000} - 5^2 = \mathbf{6.635}.$$

**6** The first digits of $157^2, 312^2, 696^2$, and $602^2$ are $\mathbf{2, 9, 4, 3}$, The sample mean is $\frac{1}{4}(2 + 9 + 4 + 3) = \frac{18}{4} = \mathbf{4.5}$. The sample variance with $N - 1 = 3$ is

$$S^2 = \frac{1}{3}\Big[(-2.5)^2 + (4.5)^2 + (-.5)^2 + (-1.5)^2\Big] = \frac{1}{3}\Big[29\Big].$$

**7** This question is about the fast way to compute $\sigma^2$ using $m^2$. The mean $m$ is probably already computed:

$$\begin{aligned} \sigma^2 &= \textstyle\sum p_i\,(x_i - m)^2 = \sum p_i\,(x_i^2 - 2mx_i + m^2) \\ &= \textstyle\sum p_i x_i^2 - 2m \sum p_i x_i + m^2 \sum p_i \\ &= \textstyle\sum p_i x_i^2 - 2m^2 + m^2 = \sum \boldsymbol{p_i x_i^2 - m^2} = \mathrm{E}\,[\boldsymbol{x^2}] - \boldsymbol{m^2}. \end{aligned}$$

**8** For $N = 24$ samples, all equal to $x = 20$,

$$\boldsymbol{\mu} = \frac{1}{N}\sum x_i = \frac{24}{24}(20) = \mathbf{20} \quad \text{and} \quad \boldsymbol{S^2} = \frac{1}{N-1}\sum (x_i - \mu)^2 = \mathbf{0}$$

For 12 samples of $x = 20$ and 12 samples of $x = 21$,

$$\boldsymbol{\mu} = \frac{12(20) + 12(21)}{24} = \mathbf{20.5} \quad \text{and} \quad \boldsymbol{S^2} = \frac{1}{N-1}\sum (x_i - \mu)^2 = \frac{1}{23}\,24\left(\frac{1}{2}\right)^2 = \frac{\mathbf{6}}{\mathbf{23}}.$$

**9** This question asks you to set up a random $0$-1 generator and run it a million times to find the average $A_{1000000}$.

One way is to use MATLAB's **rand** command with a uniform distribution between $0$ and $1$. Add $\frac{1}{2}$ to go between $0.5$ and $1.5$, then find the integer part ($0$ or $1$). Using your computed average $A_N$ (its mean is $m = \frac{1}{2}$ since $0$ and $1$ are equally likely for every sample) find the normalized variable $X$ :

$$X = \frac{A_N - \frac{1}{2}}{2\sqrt{N}} = \frac{A_N - \frac{1}{2}}{2000} \quad \text{for } N = \text{ one million.}$$

**10** The average number of heads in $N$ fair coin flips is $m = N/2$. This is obvious—but how to derive it from probabilities $p_0$ to $p_N$ of $0$ to $N$ heads? We have to compute

$$m = 0p_0 + 1p_1 + \cdots + Np_N \quad \text{with} \quad p_i = \frac{b_i}{2^N} = \frac{1}{2^N}\,\frac{N!}{i!\,(N-i)!}$$

A useful fact is $\boldsymbol{p_i = p_{N-i}}$. The probability of $\boldsymbol{i}$ **heads** equals the probability of $\boldsymbol{i}$ **tails**. If we take just those two terms in $m$, they give

$$ip_i + (N-i)p_{N-i} = ip_i + (N-i)p_i = Np_i$$

So we can compute $m$ two ways and add :

$$\begin{aligned} m &= 0p_0 + 1p_1 + \cdots + (N-1)p_{N-1} + Np_N \\ m &= Np_0 + (N-1)p_1 + \cdots + 1p_{N-1} + 0p_0 \\ 2m &= Np_0 + Np_1 + \cdots + Np_{N-1} + Np_N \\ &= N(p_0 + p_1 + \cdots + p_{N-1} + p_N) = \boldsymbol{N}. \end{aligned}$$

Then $\boldsymbol{m = N/2}$. The average number of heads is $N/2$.

**11** $\mathbf{E}\,[\boldsymbol{x^2}] = \mathrm{E}\,[(x-m)^2 + 2xm - m^2]$

$$= \mathrm{E}\,[(x-m)^2] + 2m\,\mathrm{E}\,[x] - m^2\,\mathrm{E}\,[1]$$

$$= \sigma^2 + 2m^2 - m^2 = \boldsymbol{\sigma^2 + m^2}$$

**12** The first step multiplies two independent 1-dimensional integrals (each one from $-\infty$ to $\infty$) to produce a 2-dimensional integral over the whole plane :

$$2\pi \int_{-\infty}^{\infty} p(x)\,dx \int_{-\infty}^{\infty} p(y)\,dy = 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x)\,p(y)\,dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-y^2/2}\,dxdy.$$

The second step changes to polar coordinates ($x = r\cos\theta, y = r\sin\theta, dxdy = r\,dr\,d\theta$, $x^2 + y^2 = r^2$ with $0 \le \theta \le 2\pi$ and $0 \le r \le \infty$). Notice $-x^2/2 - y^2/2 = -r^2/2$ :

$$\iint_{\text{plane}} e^{-r^2/2}\,r\,dr\,d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2/2}\,r\,dr\,d\theta$$

The $r$ and $\theta$ integrals give the answers 1 and $2\pi$ :

$$\int_{r=0}^{\infty} e^{-r^2/2}\,r\,dr = \left[-e^{-r^2/2}\right]_{r=0}^{\infty} = 1 \qquad \int_{\theta=0}^{2\pi} 1\,d\theta = 2\pi.$$

The trick was to get $e^{-r^2/2}\,r\,dr$ (which is a perfect derivative of $-e^{-r^2/2}$) by combining $e^{-x^2/2}\,dx$ and $e^{-y^2/2}\,dy$ (which can ***not*** be separately integrated from $a$ to $b$).

## Problem Set 12.2, page 554

**1** (a) Mean $m = \mathrm{E}\,[x] = (0)(1-p) + (1)(p) = p$ when the probability of heads is $p$. Here $x = 0$ for tails and $x = 1$ for heads. Notice that $0^2 = 0$ and $1^2 = 1$ so $\mathrm{E}\,[x^2] = \mathrm{E}\,[x] = p$.

$$\text{Variance} \quad \boldsymbol{\sigma^2} = \mathrm{E}\,[x^2] - m^2 = \boldsymbol{p - p^2}$$

(b) These are independent flips ! So the $N$ by $N$ covariance matrix $V$ is diagonal. The diagonal entries are the variances $\sigma^2 = p - p^2$ for each flip. Then the rule $(16-17-18)$ gives the overall variance of the sum from $N$ flips :

$$\text{overall variance } = [1 \;\; 1 \ldots 1]\, V \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = N\sigma^2 = N(p - p^2)$$

This is just saying : Add the variances for the $N$ independent experiments. Here those $N$ experiments just repeat one experiment.

**2** I am just imitating equation $(2)$ in the text. Now the experiments are numbered $3$ and $5$. They have means $m_3$ and $m_5$. The covariance $\sigma_{35}$ adds up **joint probabilities** $p_{ij}$ times (distance $x_i - m_3$) times (distance $y_j - m_5$). Here $x_i$ and $y_j$ are outputs from experiments $3$ and $5$ :

$$\boldsymbol{\sigma_{35}} = \sum_{\text{all}} \sum_{i,j} p_{ij}\, (x_i - m_3)\, (y_j - m_5).$$

**3** The 3 by 3 covariance matrix $V$ will be a sum of rank one matrices $V_{ijk} = UU^{\mathrm{T}}$ multiplied by the joint probability $p_{ijk}$ of outputs $x_i, y_j, z_k$. I am copying equation $(12)$ :

$$V = \sum_{\text{all}} \sum_{i,} \sum_{j,\ k} p_{ijk}\, UU^{\mathrm{T}} \qquad U = \begin{bmatrix} \text{output } x_i - \text{mean } \overline{x} \\ \text{output } y_j - \text{mean } \overline{y} \\ \text{output } z_k - \text{mean } \overline{z} \end{bmatrix}$$

These matrices $UU^{\mathrm{T}}$ = column times row are positive semidefinite with rank $1$ (unless $U = \mathbf{0}$). The sum $V$ is positive ***definite*** unless the $3$ experiments are dependent.

Notice that the means $\overline{x}, \overline{y}, \overline{z} = m_1, m_2, m_3$ have to be computed before the variances.

**4** We are told that the $3$ experiments are ***independent***. Then the ***covariances are zero*** off the main diagonal of $V$. This covariance matrix only shows "covariances with itself" = "variances" $\sigma_1^2, \sigma_2^2, \sigma_3^2$ on the main diagonal.

$$V = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}.$$

**5** The point is that some output $X = x_i$ must occur. So the possibilities are $Y = y_j$ and $X = x_1$, or $Y = y_j$ and $X = x_2$, or $Y = y_j$ and $X = x_3$ et cetera. The total probability of $Y = y_j$ is the sum of the conditional probabilities that $Y = y_j$ when $X = x_i$.

Here is another way to say this **law of total probability**. When $B_1, B_2, \ldots$ are separate disjoint outcomes that together account for all possible outcomes, then for any $A$

$$\text{Prob}\,(A) = \sum_i \text{Prob}\,(A \cap B_i) = \sum_i \text{Prob}\,(A|B_i)\,\text{Prob}\,(B_i).$$

**6** Prob $(A|B)$ = **conditional probability** of $A$ given $B$ satisfies this axiom:

$$\text{Prob}\,(A \textbf{\textit{ and }} B) = \text{Prob}\,(A|B)\,\text{Prob}\,(B).$$

Reason: If both $A$ and $B$ occur, then $B$ must occur—and knowing that $B$ occurs, Prob $(A|B)$ gives the probability that $A$ also occurs.

This axiom is saying that $\boldsymbol{p_{ij} = \textbf{Prob}\,(A|B)\,p_i}$

where $B$ is the event $x = x_i$ which has Prob $(B) = p_i$.

**7** The joint probabilities $p_{ij}$ = Prob $(x = x_i$ and $y = y_j)$ are in the matrix $P$:

$$P = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} \text{ with entries adding to } 1.$$

Problem **6** says that Prob $(Y = y_2|X = x_1) = \dfrac{p_{12}}{p_{11} + p_{12}} = \dfrac{0.3}{0.1 + 0.3} = \dfrac{\mathbf{3}}{\mathbf{4}}$.

Problem **5** says that Prob $(X = x_1) = p_{11} + p_{12} = 0.1 + 0.3 = \mathbf{0.4}$.

**8** This product rule of conditional probability is the axiom in Solution 12.2.6 above:

$$\text{Prob}\,(A \textbf{ and } B) = \text{Prob}\,(A \textbf{ given } B) \text{ times Prob}\,(B).$$

**9** This discussion of Bayes' Theorem is much too compressed! Let me reproduce three equations from Wolfram MathWorld. Here $A$ and $B$ are possible "sets" = "outcomes from an experiment" and the simple-looking identity $(*)$ connects conditional and unconditional probabilities.

We know from **8** that   Prob$(A$ **and** $B) =$ Prob $(A$ given $B)$ times Prob$(B)$

Reversing $A$ and $B$ gives   Prob$(A$ **and** $B) =$ Prob $(B$ given $A)$ times Prob$(A)$

$(*)$ Therefore Prob$(B$ given $A) = \dfrac{\text{Prob}\,(A \text{ given } B)\,\text{Prob}\,(B)}{\text{Prob}\,(A)}$

MathWorld gives this extension to non-overlapping sets $A_1, \ldots, A_n$ whose union is $A$:

$$\text{Prob}\,(A_i \text{ given } A) = \frac{\text{Prob}\,(A_i)\,\text{Prob}\,(A \text{ given } A_i)}{\sum_j \text{Prob}\,(A_j)\,\text{Prob}\,(A \text{ given } A_j)}$$

## Problem Set 12.3, page 560

**1** The two equations from two measurements are

$$\begin{matrix} x = b_1 \\ x = b_2 \end{matrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad \text{or} \quad A\boldsymbol{x} = \boldsymbol{b}.$$

The covariance matrix $V$ is diagonal since the measurements are independent:

$$V = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}.$$

The weighted least squares equation is $A^{\mathrm{T}} V^{-1} A\widehat{\boldsymbol{x}} = A^{\mathrm{T}} V^{-1}\boldsymbol{b}$.

$$A^{\mathrm{T}} V^{-1} A = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sigma_1^2 & 0 \\ 0 & 1/\sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

$$A^{\mathrm{T}} V^{-1} \boldsymbol{b} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sigma_1^2 & 0 \\ 0 & 1/\sigma_2^2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{b_1}{\sigma_1^2} + \frac{b_2}{\sigma_2^2}$$

Then $\widehat{\boldsymbol{x}}$ is the ratio of those numbers:

$$\widehat{\boldsymbol{x}} = \frac{b_1/\sigma_1^2 + b_2/\sigma_2^2}{1/\sigma_1^2 + 1/\sigma_2^2}$$

The variance of that estimate $\widehat{x}$ should be written as in $(13)$:

$$\mathrm{E}\left[(\widehat{\boldsymbol{x}}-\boldsymbol{x})\,(\widehat{\boldsymbol{x}}-\boldsymbol{x})^{\mathrm{T}}\right]=(A^{\mathrm{T}}\,V^{-1}\,A)^{-1}=\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)^{-1}.$$

**2** (a) In the limit $\sigma_2\to 0$ the ratio $\widehat{\boldsymbol{x}}$ approaches $b_2$ because:

$$(\text{Multiply } \widehat{\boldsymbol{x}} \text{ above and below by } \sigma_1^2\,\sigma_2^2)\quad \widehat{\boldsymbol{x}}=\frac{b_1\sigma_2^2+b_2\sigma_1^2}{\sigma_2^2\;+\;\sigma_1^2}\to\frac{b_2\sigma_1^2}{\sigma_1^2}=\boldsymbol{b_2}.$$

The second equation $x=b_2$ is 100% accurate if its variance is $\sigma_2=0$.

(b) If $\sigma_2\to\infty$ then $1/\sigma_2^2\to 0$ and $\widehat{\boldsymbol{x}}\to\dfrac{b_1/\sigma_1^2}{1/\sigma_1^2}=\boldsymbol{b_1}$. We are getting *no information* from the totally unreliable measurement $x=b_2$.

**3** The key fact of **independence** is in the equation $p(x,y)=p(x)\,p(y)$. Then

$$\iint p(x,y)\,dx\,dy=\iint p(x)\,p(y)\,dx\,dy=\int p(x)\,dx\int p(y)\,dy=(1)\,(1)=\mathbf{1}.$$

$$\begin{aligned}\iint(x+y)\,p(x,y)\,dx\,dy &=\iint x\,p(x)\,p(y)\,dx\,dy+\iint y\,p(x)\,p(y)\,dx\,dy\\ &=\int x\,p(x)\,dx\int p(y)\,dy+\int p(x)\,dx\int y\,p(y)\,dy\\ &=(m_x)\,(1)+(1)\,(m_y)=m_x+m_y.\end{aligned}$$

**4** Continue Problem 3 to find variances $\sigma_x^2$ and $\sigma_y^2$ and to see covariance $\sigma_{xy}=0$:

$$\iint(x-m_x)^2\,p(x,y)\,dx\,dy=\int(x-m_x)^2\,p(x)\,dx\int p(y)\,dy=\boldsymbol{\sigma_x^2}$$

$$\iint(x-m_x)\,(y-m_y)\,p(x,y)\,dx\,dy=\int(x-m_x)\,p(x)\,dx\int(y-m_y)\,p(y)\,dy=(0)\,(0).$$

**5** We are inverting a 2 by 2 matrix using $\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$:

$$V^{-1}=\begin{bmatrix}\sigma_1^2&\sigma_{12}\\\sigma_{12}&\sigma_2^2\end{bmatrix}^{-1}=\frac{1}{\sigma_1^2\,\sigma_2^2-\sigma_{12}^2}\begin{bmatrix}\sigma_2^2&-\sigma_{12}\\-\sigma_{12}&\sigma_1^2\end{bmatrix}=\qquad\qquad \rho=\frac{\sigma_{12}}{\sigma_1\,\sigma_2}$$

$$\frac{1}{\sigma_1^2\,\sigma_2^2(1-\rho^2)}\begin{bmatrix}\sigma_2^2&-\sigma_{12}\\-\sigma_{12}&\sigma_1^2\end{bmatrix}=\frac{1}{1-\rho^2}\begin{bmatrix}1/\sigma_1^2&-\rho/\sigma_1\,\sigma_2\\-\rho/\sigma_1\,\sigma_2&1/\sigma_2^2\end{bmatrix}$$

**6** The right hand side of $\widehat{x}_{k+1}$ shows the **gain factor** $\mathbf{1/(k+1)}$ :

$$\widehat{x}_k+\frac{1}{k+1}(b_{k+1}-\widehat{x}_k)=\frac{b_1+\cdots+b_k}{k}+\frac{\mathbf{1}}{\boldsymbol{k+1}}\left(b_{k+1}-\frac{b_1+\cdots+b_k}{k}\right)=\frac{b_1+\cdots+b_{k+1}}{k+1}$$

Check that each number $b_1, b_2, \ldots, b_k, b_{k+1}$ is correctly divided by $k+1$ :

$$\frac{1}{k}-\frac{1}{k+1}\,\frac{1}{k}=\frac{1}{k}\left(\frac{k+1\ \ -1}{k+1}\right)=\frac{1}{k+1}.$$

**7** We are considering the case when all the measurements $b_1, b_2, \ldots, b_{k+1}$ have the same variance $\sigma^2$. We know that the correct variance of their average is $W_{k+1}=\sigma^2/(k+1)$. We want to see how this answer comes from equation $(18)$ when we have the correct $W_k=\sigma^2/k$ from the previous step, and we update to $W_{k+1}$ :

$(18)$ says that $W_{k+1}^{-1}=W_k^{-1}+A_{k+1}^{\mathrm{T}}\,V_{k+1}^{-1}\,A_{k+1}=\dfrac{k}{\sigma^2}+[1]\,[1/\sigma^2]\,[1]=\dfrac{k}{\sigma^2}+\dfrac{1}{\sigma^2}=\dfrac{k+1}{\sigma^2}$.

So $W_{k+1}=\sigma^2/(k+1)$ is correct at the new step (and forever by induction).

**8** The three equations have variances $\sigma^2, s^2, \sigma^2$ and they have ***zero covariances***. (This makes the step-by-step Kalman filter possible.) We can divide the equations by $\sigma, s, \sigma$ to produce ***unit variances*** (which lead to ordinary unweighted least squares). We are given $F=1$ :

$$\begin{bmatrix} 1/\sigma & 0 \\ -1/s & 1/s \\ 0 & 1/\sigma \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}=\begin{bmatrix} b_0/\sigma \\ 0 \\ b_1/\sigma \end{bmatrix} \text{ is our } A\boldsymbol{x}=\boldsymbol{b}.$$

The normal equation (now unweighted) is $A^{\mathrm{T}}A\,\widehat{\boldsymbol{x}}=A^{\mathrm{T}}\boldsymbol{b}$:

$$\begin{bmatrix} \dfrac{1}{\sigma^2}+\dfrac{1}{s^2} & -\dfrac{1}{s^2} \\ -\dfrac{1}{s^2} & \dfrac{1}{\sigma^2}+\dfrac{1}{s^2} \end{bmatrix}\begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{bmatrix}=\begin{bmatrix} \dfrac{b_0}{\sigma^2} \\ \dfrac{b_1}{\sigma^2} \end{bmatrix}.$$

The determinant of this $A^{\mathrm{T}}A$ is $\det=\dfrac{1}{\sigma^4}+\dfrac{2}{\sigma^2 s^2}$. The solution is

$$\widehat{x}_1=\frac{1}{\det}\left(\frac{b_0}{\sigma^4}+\frac{b_0}{\sigma^2 s^2}+\frac{b_1}{\sigma^2 s^2}\right)\qquad \widehat{x}_2=\frac{1}{\det}\left(\frac{b_0}{\sigma^2 s^2}+\frac{b_1}{\sigma^2 s^2}+\frac{b_1}{\sigma^4}\right).$$

**9** With $A = I$ and $\boldsymbol{u}^{\mathrm{T}} = \boldsymbol{v}^{\mathrm{T}} = [1\ \ 2\ \ 3]$ we can use the direct formula for $M^{-1}$ :

$$(I - \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})^{-1} = I + \frac{\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}}{1 - \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}} = I + \frac{1}{1 - 14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{1}{13} & \frac{2}{13} & \frac{3}{13} \\ \frac{2}{13} & 1 - \frac{4}{13} & \frac{6}{13} \\ \frac{3}{13} & \frac{6}{13} & 1 - \frac{9}{13} \end{bmatrix}. \text{ Multiply } \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \text{ to get } \boldsymbol{y} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - \frac{16}{13}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 10 \\ -19 \\ 4 \end{bmatrix}.$$

Instead of this formula for $(I = \boldsymbol{u}\,\boldsymbol{v}^{\mathrm{T}})^{-1}$, try steps 1 and 2 :

**Step 1** with $A = I$ gives $\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{z} = \boldsymbol{u}$.

**Step 2** gives $\boldsymbol{y} = \boldsymbol{b} - \dfrac{\boldsymbol{v}^{\mathrm{T}}\,\boldsymbol{u}}{13}\,\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - \dfrac{16}{13}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as before.

**10** We are asked to check that $M\boldsymbol{y} = \boldsymbol{b}$ using the update formula. Start with

$$M\boldsymbol{y} = (A - \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})\left(\boldsymbol{x} + \frac{\boldsymbol{v}^{\mathrm{T}}\,\boldsymbol{x}}{c}\,\boldsymbol{z}\right)$$

$$= A\boldsymbol{x} - \boldsymbol{u}\,(\boldsymbol{v}^{\mathrm{T}}\,\boldsymbol{x}) + \frac{\boldsymbol{v}^{\mathrm{T}}\,\boldsymbol{x}\,A\boldsymbol{z}}{c} - \frac{\boldsymbol{u}\,(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{z})\,(\boldsymbol{v}^{\mathrm{T}}\,\boldsymbol{x})}{c}$$

Since $A\boldsymbol{x} = \boldsymbol{b}$ we hope the other 3 terms combine to give zero when $A\boldsymbol{z} = \boldsymbol{u}$

$$\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\,\boldsymbol{x}\left[-1 + \frac{1}{c} - \frac{\boldsymbol{v}^{\mathrm{T}}\boldsymbol{z}}{c}\right] = \frac{\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\,\boldsymbol{x}}{c}\left[-c + 1 - \boldsymbol{v}^{\mathrm{T}}\,\boldsymbol{z}\right] = \mathbf{0} \text{ from the formula for } c$$

**11** Multiply **columns times rows** to see that the new $\boldsymbol{v}$ changes $A^{\mathrm{T}}A$ to

$$\begin{bmatrix} A^{\mathrm{T}} & \boldsymbol{v} \end{bmatrix}\begin{bmatrix} A \\ \boldsymbol{v}^{\mathrm{T}} \end{bmatrix} = A^{\mathrm{T}}A + \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}$$

So adding the new row to $A$ (and of course the new column to $A^{\mathrm{T}}$) has increased $A^{\mathrm{T}}\,A$ by the rank one matrix $\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}$.

The book is ending with matrix multiplication! We could allow changes of rank $r$ :

When $A$ changes to $M = A - UW^{-1}V$, its inverse changes to

$$\boldsymbol{M^{-1}} = A^{-1} + A^{-1}\,U(W - VA^{-1}\,U)^{-1}\,VA^{-1}.$$

This change has rank $r$ when $W_{r\times r}$ and $V_{r\times n}$ and $U_{n\times r}$ all have rank $r$.