INTRODUCTION

TO

LINEAR

ALGEBRA

Fifth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 9.1, page 436

1 (a)(b)(c) have sums 4, $-2+2i$, $2\cos\theta$ and products 5, $-2i$, 1. Note $(e^{i\theta})(e^{-i\theta}) = 1$.

2 In polar form these are $\sqrt{5}e^{i\theta}$, $5e^{2i\theta}$, $\frac{1}{\sqrt{5}}e^{-i\theta}$, $\sqrt{5}$.

3 The absolute values are $r = 10$, 100, $\frac{1}{10}$, and 100. The angles are θ, 2θ, $-\theta$ and -2θ.

4 $|z \times w| = 6$, $|z+w| \le 5$, $|z/w| = \frac{2}{3}$, $|z-w| \le 5$.

5 $a + ib = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, i, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$; $w^{12} = 1$.

6 $1/z$ has absolute value $1/r$ and angle $-\theta$; $(1/r)e^{-i\theta}$ times $re^{i\theta}$ equals 1.

7 $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} ac-bd \\ bc+ad \end{bmatrix}$ **real part** / **imaginary part** $\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$ is the matrix form of $(1+3i)(1-3i) = 10$.

8 $\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}$ gives complex matrix = vector multiplication $(A_1 + iA_2)(\boldsymbol{x}_1 + i\boldsymbol{x}_2) = \boldsymbol{b}_1 + i\boldsymbol{b}_2$.

9 $2+i$; $(2+i)(1+i) = 1+3i$; $e^{-i\pi/2} = -i$; $e^{-i\pi} = -1$; $\frac{1-i}{1+i} = -i$; $(-i)^{103} = i$.

10 $z + \overline{z}$ is real; $z - \overline{z}$ is pure imaginary; $z\overline{z}$ is positive; $z/\overline{z}$ has absolute value 1.

11 $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ includes aI (which just adds a to the eigenvalues and $b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. So the eigenvectors are $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$. The eigenvalues are $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$. We see $\overline{\boldsymbol{x}}_1 = \boldsymbol{x}_2$ and $\overline{\lambda}_1 = \lambda_2$ as expected for real matrices with complex eigenvalues.

12 (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; λ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a + d$ when $ad = bc$ (c) the λ's can be real and different.

13 Complex λ's when $(a+d)^2 < 4(ad-bc)$; write $(a+d)^2 - 4(ad-bc)$ as $(a-d)^2 + 4bc$ which is positive when $bc > 0$.

14 The symmetric block matrix has real eigenvalues; so $i\lambda$ is real and λ is pure imaginary.

15 (a) $2e^{i\pi/3}$, $4e^{2i\pi/3}$ (b) $e^{2i\theta}$, $e^{4i\theta}$ (c) $7e^{3\pi i/2}$, $49e^{3\pi i}$ $(= -49)$ (d) $\sqrt{50}e^{-\pi i/4}$, $50e^{-\pi i/2}$.

16 $r = 1$, angle $\frac{\pi}{2} - \theta$; multiply by $e^{i\theta}$ to get $e^{i\pi/2} = i$.

17 $a + ib = 1, i, -1, -i, \pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$. The root $\overline{w} = w^{-1} = e^{-2\pi i/8}$ is $1/\sqrt{2} - i/\sqrt{2}$.

18 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$ are cube roots of 1. The cube roots of -1 are -1, $e^{\pi i/3}$, $e^{-\pi i/3}$. Altogether six roots of $z^6 = 1$.

19 $\cos 3\theta = \text{Re}[(\cos\,\theta + i \sin\theta)^3] = \cos^3\theta - 3\cos\theta\sin^2\theta$; $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

20 If the conjugate $\overline{z} = 1/z$ then $|z|^2 = 1$ and z is any point $e^{i\theta}$ on the unit circle.

21 e^i is at angle $\theta = 1$ on the unit circle; $|i^e| = 1^e$; Infinitely many $i^e = e^{i(\pi/2+2\pi n)e}$.

22 (a) Unit circle (b) Spiral in to $e^{-2\pi}$ (c) Circle continuing around to angle $\theta = 2\pi^2$.

Problem Set 9.2, page 443

1 $\|\boldsymbol{u}\| = \sqrt{9} = 3$, $\|\boldsymbol{v}\| = \sqrt{3}$, $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = 3i + 2$, $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u} = -3i + 2$ (this is the conjugate of $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$).

2 $A^{\mathrm{H}}A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix}$ and $AA^{\mathrm{H}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ are Hermitian matrices. They share the eigenvalues 4 and 2.

3 $\boldsymbol{z} =$ multiple of $(1+i, 1+i, -2)$; $A\boldsymbol{z} = \boldsymbol{0}$ gives $\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}} = \boldsymbol{0}^{\mathrm{H}}$ so $\boldsymbol{z}$ (not $\overline{\boldsymbol{z}}$!) is orthogonal to all columns of A^{H} (using complex inner product $\boldsymbol{z}^{\mathrm{H}}$ times columns of A^{H}).

4 The four fundamental subspaces are now $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$, $\boldsymbol{C}(A^{\mathrm{H}})$, $\boldsymbol{N}(A^{\mathrm{H}})$. A^{H} ***and not*** A^{T}.

5 (a) $(A^{\mathrm{H}}A)^{\mathrm{H}} = A^{\mathrm{H}}A^{\mathrm{HH}} = A^{\mathrm{H}}A$ again (b) If $A^{\mathrm{H}}A\boldsymbol{z} = \boldsymbol{0}$ then $(\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}})(A\boldsymbol{z}) = 0$. This is $\|A\boldsymbol{z}\|^2 = 0$ so $A\boldsymbol{z} = \boldsymbol{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are always the ***same***.

6 (a) False (c) False $A = Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) True: $-i$ is not an eigenvalue when $S = S^{\mathrm{H}}$.

7 cS is still Hermitian ***for real*** c; $(iS)^{\mathrm{H}} = -iS^{\mathrm{H}} = -iS$ is skew-Hermitian.

8 This P is invertible and unitary. $P^2 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, $P^3 = \begin{bmatrix} -i & & \\ & -i & \\ & & -i \end{bmatrix} = -iI$. Then $P^{100} = (-i)^{33}P = -iP$. The eigenvalues of P are the roots of $\lambda^3 = -i$, which are i and $ie^{2\pi i/3}$ and $ie^{4\pi i/3}$.

9 One unit eigenvector is certainly $\boldsymbol{x}_1 = (1, 1, 1)$ with $\lambda_1 = i$. The other eigenvectors are $\boldsymbol{x}_2 = (1, w, w^2)$ and $\boldsymbol{x}_3 = (1, w^2, w^4)$ with $w = e^{2\pi i/3}$. ***The eigenvector matrix is the Fourier matrix*** F_3. The eigenvectors of any unitary matrix like P are orthogonal (using the correct complex form $\boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}$ of the inner product).

10 $(1, 1, 1)$, $(1, e^{2\pi i/3}, e^{4\pi i/3})$, $(1, e^{4\pi i/3}, e^{2\pi i/3})$ are orthogonal (complex inner product!) because P is an orthogonal matrix—and therefore its eigenvector matrix is unitary.

11 If $Q^{\mathrm{H}}Q = I$ then $Q^{-1}(Q^{\mathrm{H}})^{-1} = Q^{-1}(Q^{-1})^{\mathrm{H}} = I$ so Q^{-1} is also unitary. Also $(QU)^{\mathrm{H}}(QU) = U^{\mathrm{H}}Q^{\mathrm{H}}QU = U^{\mathrm{H}}U = I$ so QU is unitary.

12 Determinant = product of the eigenvalues ***(all real)***. And $A = A^{\mathrm{H}}$ gives $\det A = \overline{\det A}$.

13 $(\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}})(A\boldsymbol{z}) = \|A\boldsymbol{z}\|^2$ is positive unless $A\boldsymbol{z} = \mathbf{0}$. When A has independent columns this means $\boldsymbol{z} = \mathbf{0}$; so $A^{\mathrm{H}}A$ is positive definite.

14 $S = \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$.

15 $K =$(iA^{T} in Problem 14)$= \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix}\begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix}\dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}$; λ's are imaginary.

16 $U = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ has $|\lambda| = 1$.

17 $U = \dfrac{1}{L}\begin{bmatrix} 1+\sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\dfrac{1}{L}\begin{bmatrix} 1+\sqrt{3} & 1-i \\ -1-i & 1+\sqrt{3} \end{bmatrix}$ with $L^2 = 6+2\sqrt{3}$. Unitary means $|\lambda| = 1$. $U = U^{\mathrm{H}}$ gives real λ. Then trace zero gives $\lambda = 1$ and -1.

18 The $\boldsymbol{v}$'s are columns of a unitary matrix U, so U^{H} is U^{-1}. Then $\boldsymbol{z} = UU^{\mathrm{H}}\boldsymbol{z} =$ (multiply by columns) $= \boldsymbol{v}_1(\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{z}) + \cdots + \boldsymbol{v}_n(\boldsymbol{v}_n^{\mathrm{H}}\boldsymbol{z})$: a typical orthonormal expansion.

19 $\boldsymbol{z} = (1, i, -2)$ completes an orthogonal basis for $\mathbf{C}^3$. So does any $e^{i\theta}\boldsymbol{z}$.

20 $S = A + iB = (A + iB)^{\mathrm{H}} = A^{\mathrm{T}} - iB^{\mathrm{T}};$ A is symmetric but B is skew-symmetric.

21 $\mathbf{C}^n$ has dimension n; the columns of any unitary matrix are a basis. For example use the columns of iI: $(i, 0, \ldots, 0), \ldots, (0, \ldots, 0, i)$

22 $[\,1\,]$ and $[\,-1\,]$; any $[\,e^{i\theta}\,]$; $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$; $\begin{bmatrix} w & e^{i\phi}\overline{z} \\ -z & e^{i\phi}\overline{w} \end{bmatrix}$ with $|w|^2 + |z|^2 = 1$ and any angle ϕ

23 The eigenvalues of A^{H} are ***complex conjugates*** of the eigenvalues of A: $\det(A - \lambda I) = 0$ gives $\det(A^{\mathrm{H}} - \overline{\lambda} I) = 0$.

24 $(I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}})^{\mathrm{H}} = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ and also $(I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}})^2 = I - 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}} + 4\boldsymbol{u}(\boldsymbol{u}^{\mathrm{H}}\boldsymbol{u})\boldsymbol{u}^{\mathrm{H}} = I$. The rank-1 matrix $\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ projects onto the line through $\boldsymbol{u}$.

25 Unitary $U^{\mathrm{H}}U = I$ means $(A^{\mathrm{T}} - iB^{\mathrm{T}})(A + iB) = (A^{\mathrm{T}}A + B^{\mathrm{T}}B) + i(A^{\mathrm{T}}B - B^{\mathrm{T}}A) = I$.

$A^{\mathrm{T}}A + B^{\mathrm{T}}B = I$ and $A^{\mathrm{T}}B - B^{\mathrm{T}}A = 0$ which makes the block matrix orthogonal.

26 We are given $A + iB = (A + iB)^{\mathrm{H}} = A^{\mathrm{T}} - iB^{\mathrm{T}}$. Then $A = A^{\mathrm{T}}$ and $B = -B^{\mathrm{T}}$. So that $\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix}$ is symmetric.

27 $SS^{-1} = I$ gives $(S^{-1})^{\mathrm{H}}S^{\mathrm{H}} = I$. Therefore $(S^{-1})^{\mathrm{H}}$ is $(S^{\mathrm{H}})^{-1} = S^{-1}$ and S^{-1} is Hermitian.

28 If U has (complex) orthonormal columns, then $U^{\mathrm{H}}U = I$ and U is ***unitary***. If those columns are eigenvectors of A, then $A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ is ***normal***. The direct test for a normal matrix (which is $AA^{\mathrm{H}} = A^{\mathrm{H}}A$ because diagonals could be real!) and Λ^{H} surely commute:

$$AA^{\mathrm{H}} = (U\Lambda U^{\mathrm{H}})(U\Lambda^{\mathrm{H}}U^{\mathrm{H}}) = U(\Lambda\Lambda^{\mathrm{H}})U^{\mathrm{H}} = U(\Lambda^{\mathrm{H}}\Lambda)U^{\mathrm{H}} = (U\Lambda^{\mathrm{H}}U^{\mathrm{H}})(U\Lambda U^{\mathrm{H}}) = A^{\mathrm{H}}A.$$

An easy way to construct a normal matrix is $1 + i$ times a symmetric matrix. Or take $A = S + iT$ where the real symmetric S and T commute (Then $A^{\mathrm{H}} = S - iT$ and $AA^{\mathrm{H}} = A^{\mathrm{H}}A$).

Problem Set 9.3, page 450

1 Equation (3) (the FFT) is correct using $i^2 = -1$ in the last two rows and three columns.

2 $F^{-1} = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & -i & & i \end{bmatrix} = \frac{1}{4} F^{\mathrm{H}}.$

3 $F = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & -i & & i \end{bmatrix}$ permutation last.

4 $D = \begin{bmatrix} 1 & & \\ & e^{2\pi i/6} & \\ & & e^{4\pi i/6} \end{bmatrix}$ (note 6 not 3) and $F_3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}$.

5 $F^{-1}\boldsymbol{w} = \boldsymbol{v}$ and $F^{-1}\boldsymbol{v} = \boldsymbol{w}/4$. Delta vector $\leftrightarrow$ all-ones vector.

6 $(F_4)^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$ and $(F_4)^4 = 16I$. Four transforms recover the signal!

7 $\boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = F\boldsymbol{c}$. Also $C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} = FC$.
Adding $c + C$ gives $(1, 1, 1, 1)$ to $(4, 0, 0, 0) = 4$ (delta vector).

8 $\boldsymbol{c} \to (1, 1, 1, 1, 0, 0, 0, 0) \to (4, 0, 0, 0, 0, 0, 0, 0) \to (4, 0, 0, 0, 4, 0, 0, 0) = F_8\boldsymbol{c}$.
$\boldsymbol{C} \to (0, 0, 0, 0, 1, 1, 1, 1) \to (0, 0, 0, 0, 4, 0, 0, 0) \to (4, 0, 0, 0, -4, 0, 0, 0) = F_8\boldsymbol{C}$.

9 If $w^{64} = 1$ then w^2 is a 32nd root of 1 and $\sqrt{w}$ is a 128th root of 1: Key to FFT.

10 For every integer n, the nth roots of 1 add to zero. For even n, they cancel in pairs. For any n, use the geometric series formula $1 + w + \cdots + w^{n-1} = (w^n - 1)/(w - 1) = 0$. In particular for $n = 3$, $1 + (-1 + i\sqrt{3})/2 + (-1 - i\sqrt{3})/2 = 0$.

11 The eigenvalues of P are $1, i, i^2 = -1$, and $i^3 = -i$. Problem 11 displays the eigenvectors. And also $\det(P - \lambda I) = \lambda^4 - 1$.

12 $\Lambda = \text{diag}(1, i, i^2, i^3)$; $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and P^{T} lead to $\lambda^3 - 1 = 0$.

13 $e_1 = c_0 + c_1 + c_2 + c_3$ and $e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3$; E contains the four eigenvalues of $C = FEF^{-1}$ because F contains the eigenvectors.

14 Eigenvalues $e_1 = 2 - 1 - 1 = 0$, $e_2 = 2 - i - i^3 = 2$, $e_3 = 2 - (-1) - (-1) = 4$, $e_4 = 2 - i^3 - i^9 = 2$. Just transform column 0 of C. Check trace $0 + 2 + 4 + 2 = 8$.

15 Diagonal E needs n multiplications, Fourier matrix F and F^{-1} need $\frac{1}{2} n \log_2 n$ multiplications each by the **FFT**. The total is much less than the ordinary n^2 for C times $\boldsymbol{x}$.

16 The row $1, \overline{w}^k, \overline{w}^{2k}, \ldots$ in $\overline{F}$ is the same as the row $1, w^{N-k}, w^{N-2k}, \ldots$ in F because $w^{N-k} = e^{(2\pi i/N)(N-k)}$ is $e^{2\pi i} e^{-(2\pi i/N)k} = 1$ times $\overline{w}^k$. So F and $\overline{F}$ have the **same rows in reversed order** (except for row 0 which is all ones).

17

0	000 reverses to $000 = \mathbf{0}$		
1	001 reverses to $100 = \mathbf{4}$		
2	010 reverses to $010 = \mathbf{2}$	**Now evens come before odds**!
3	011 reverses to $110 = \mathbf{6}$		
4	100 reverses to $001 = \mathbf{1}$		
5	101 reverses to $101 = \mathbf{5}$		
6	110 reverses to $011 = \mathbf{3}$		
7	111 reverses to $111 = \mathbf{7}$		