INTRODUCTION

TO

LINEAR

ALGEBRA

Fifth Edition

MANUAL FOR INSTRUCTORS

Gilbert Strang
Massachusetts Institute of Technology

math.mit.edu/linearalgebra

web.mit.edu/18.06

video lectures: ocw.mit.edu

math.mit.edu/~gs

www.wellesleycambridge.com

email: linearalgebrabook@gmail.com

Wellesley - Cambridge Press

Box 812060

Wellesley, Massachusetts 02482
Problem Set 5.1, page 254

1 \( \det(2A) = 2^4 \det A = 8; \) \( \det(-A) = (-1)^4 \det A = \frac{1}{2}; \) \( \det(A^2) = \frac{1}{4}; \) \( \det(A^{-1}) = 2. \)

2 \( \det\left(\frac{1}{2}A\right) = \left(\frac{1}{2}\right)^3 \det A = -\frac{1}{8} \) and \( \det(-A) = (-1)^3 \det A = 1; \) \( \det(A^2) = 1; \) \( \det(A^{-1}) = -1. \)

3 (a) False: \( \det(I + I) \) is not \( 1 + 1 \) (except when \( n = 1 \)) (b) True: The product rule extends to \( ABC \) (use it twice) (c) False: \( \det(4A) \) is \( 4^n \det A \) (d) False: \( A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) is invertible.

4 Exchange rows 1 and 3 to show \( |J_3| = -1. \) Exchange rows 1 and 4, then rows 2 and 3 to show \( |J_4| = 1. \)

5 \( |J_5| = 1 \) by exchanging row 1 with 5 and row 2 with 4. \( |J_6| = -1, \) \( |J_7| = -1. \) Determinants 1, 1, -1, -1 repeat in cycles of length 4 so the determinant of \( J_{101} \) is +1.

6 To prove Rule 6, multiply the zero row by \( t = 2. \) The determinant is multiplied by 2 (Rule 3) but the matrix is the same. So \( 2 \det(A) = \det(A) \) and \( \det(A) = 0. \)

7 \( \det(Q) = 1 \) for rotation and \( \det(Q) = 1 - 2 \sin^2 \theta - 2 \cos^2 \theta = -1 \) for reflection.

8 \( Q^TQ = I \Rightarrow |Q^T||Q| = |Q|^2 = 1 \Rightarrow |Q| = \pm 1; \) \( Q^n \) stays orthogonal so its determinant can’t blow up as \( n \to \infty. \)

9 \( \det A = 1 \) from two row exchanges. \( \det B = 2 \) (subtract rows 1 and 2 from row 3, then columns 1 and 2 from column 3). \( \det C = 0 \) (equal rows) even though \( C = A + B! \)

10 If the entries in every row add to zero, then \((1, 1, \ldots, 1)\) is in the nullspace: singular \( A \) has \( \det = 0. \) (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of \( A - I \) add to zero (not necessarily \( \det A = 1). \)

11 \( CD = -DC \Rightarrow \det CD = (-1)^n \det DC \) and not just \( -\det DC. \) If \( n \) is even then \( \det CD = \det DC \) and we can have an invertible \( CD. \)

12 \( \det(A^{-1}) \) divides twice by \( ad - bc \) (once for each row). This gives \( \det A^{-1} = \frac{ad - bc}{(ad - bc)^2} = \frac{1}{ad - bc}. \)
13 Pivots 1, 1, 1 give determinant = 1; pivots 1, –2, –3/2 give determinant = 3.

14 det(A) = 36 and the 4 by 4 second difference matrix has det = 5.

15 The first determinant is 0, the second is \(1 - 2t^2 + t^4 = (1 - t^2)^2\).

16 A singular rank one matrix has determinant = 0. The skew-symmetric \(K\) also has det \(K = 0\) (see #17): a skew-symmetric matrix \(K\) of odd order 3.

17 Any 3 by 3 skew-symmetric \(K\) has det\((K^T) = \) det\((-K) = (-1)^3\)det\((K)\). This is \(-\)det\((K)\). But always det\((K^T) = \) det\((K)\). So we must have det\((K) = 0\) for 3 by 3.

\[
\begin{vmatrix}
1 & a & a^2 \\
1 & b & b^2 \\
1 & c & c^2 \\
\end{vmatrix}
= \begin{vmatrix}
1 & a & a^2 \\
0 & b-a & b^2-a^2 \\
0 & c-a & c^2-a^2 \\
\end{vmatrix}
= \begin{vmatrix}
b-a & b^2-a^2 \\
c-a & c^2-a^2 \\
\end{vmatrix}
\text{ (to reach 2 by 2, eliminate } a \text{ and } a^2 \text{ in row 1 by column operations)—subtract } a \text{ and } a^2 \text{ times column 1 from columns 2 and 3. Factor out } b-a \text{ and } c-a \text{ from the 2 by 2:}
\]

\[
(b-a)(c-a) \begin{vmatrix}
1 & b+a \\
1 & c+a \\
\end{vmatrix} = (b-a)(c-a)(c-b).
\]

18 For triangular matrices, just multiply the diagonal entries: det\((U) = 6\), det\((U^{-1}) = \frac{1}{6}\).

and det\((U^2) = 36\). 2 by 2 matrix: det\((U) = ad\), det\((U^2) = a^2d^2\). If \(ad \neq 0\) then det\((U^{-1}) = \frac{1}{ad}\).

19 det \[
\begin{vmatrix}
a-\ell c & b-\ell d \\
c-\ell a & d-\ell b \\
\end{vmatrix}
\]
reduces to \((ad - bc)(1 - \ell)\). The determinant changes if you do two row operations at once.

20 We can exchange rows using the three elimination steps in the problem, followed by multiplying row 1 by \(-1\). So Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

21 det\((A) = 3\), det\((A^{-1}) = \frac{1}{3}\), det\((A - \lambda I) = \lambda^2 - 4\lambda + 3\). The numbers \(\lambda = 1\) and \(\lambda = 3\) give det\((A - \lambda I) = 0\). The (singular!) matrices are

\[
A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}
\]
Note to instructor: You could explain that this is the reason determinants come before eigenvalues. Identify \( \lambda = 1 \) and \( \lambda = 3 \) as the eigenvalues of \( A \).

23 \( A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \) has \( \det(A) = 10, \ A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}, \ \det(A^2) = 100, \ A^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} \) has \( \det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0 \) when \( \lambda = 2 \) or 5; those are eigenvalues.

24 Here \( A = LU \) with \( \det(L) = 1 \) and \( \det(U) = -6 = \) product of pivots, so also \( \det(A) = -6. \ \det(U^{-1}L^{-1}) = -\frac{1}{6} = 1/\det(A) \) and \( \det(U^{-1}L^{-1}A) \) is \( I = 1. \)

25 When the \( i, j \) entry is \( ij \), row 2 = 2 times row 1 so \( \det A = 0. \)

26 When the \( ij \) entry is \( i + j \), row 3 − row 2 = row2 − row 1 so \( A \) is singular: \( \det A = 0. \)

27 \( \det A = abc, \ \det B = -abcd, \ \det C = a(b-a)(c-b) \) by doing elimination.

28 (a) True: \( \det(AB) = \det(A) \det(B) = 0 \) (b) False: A row exchange gives \( - \det = \) product of pivots. (c) False: \( A = 2I \) and \( B = I \) have \( A-B = I \) but the determinants have \( 2^n - 1 \neq 1 \) (d) True: \( \det(AB) = \det(A) \det(B) = \det(BA). \)

29 \( A \) is rectangular so \( \det(A^T A) \neq (\det A^T)(\det A): \) these determinants are not defined. In fact if \( A \) is tall and thin \( (m > n) \), then \( \det(A^T A) \) adds up \( |\det B|^2 \) where the \( B \)'s are all the \( n \) by \( n \) submatrices of \( A. \)

30 Derivatives of \( f = \ln(ad - bc): \)

\[
\begin{bmatrix}
\frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\
\frac{\partial f}{\partial b} & \frac{\partial f}{\partial d}
\end{bmatrix} = \begin{bmatrix}
d & -b \\
ad - bc & ad - bc
\end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix}
d & -b \\
ad - bc & ad - bc
\end{bmatrix} = A^{-1}.
\]

31 The Hilbert determinants are \( 1, \ 8 \times 10^{-2}, \ 4.6 \times 10^{-4}, \ 1.6 \times 10^{-7}, \ 3.7 \times 10^{-12}, \ 5.4 \times 10^{-18}, \ 4.8 \times 10^{-25}, \ 2.7 \times 10^{-33}, \ 9.7 \times 10^{-43}, \ 2.2 \times 10^{-53}. \) Pivots are ratios of determinants so the 10th pivot is near \( 10^{-10}. \) The Hilbert matrix is numerically difficult \( (ill-conditioned). \) Please see the Figure 7.1 and Section 8.3.
Typical determinants of \texttt{rand}(n) are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. \texttt{randn}(n) with normal distribution gives $10^{31}, 10^{78}, 10^{186}, \text{Inf}$ which means $\geq 2^{1024}$. MATLAB allows $1.99999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives \texttt{Inf}!

I now know that maximizing the determinant for 1, $-1$ matrices is Hadamard’s problem (1893): see Brenner in American Math. Monthly volume 79 (1972) 626-630. Neil Sloane’s wonderful On-Line Encyclopedia of Integer Sequences (\texttt{research.att.com/~njas}) includes the solution for small $n$ (and more references) when the problem is changed to 0, 1 matrices. That sequence A003432 starts from $n = 0$ with 1, 1, 1, 2, 3, 5, 9. Then the 1, $-1$ maximum for size $n$ is $2^{n-1}$ times the 0, 1 maximum for size $n - 1$ (so (32)(5) = 160 for $n = 6$ in sequence A003433).

To reduce the 1, $-1$ problem from 6 by 6 to the 0, 1 problem for 5 by 5, multiply the six rows by $\pm 1$ to put +1 in column 1. Then subtract row 1 from rows 2 to 6 to get a 5 by 5 submatrix $S$ with entries $-2$ and 0. Then divide $S$ by $-2$.

Here is an advanced MATLAB code that finds a 1, $-1$ matrix with largest $\det A = 48$ for $n = 5$:

```
n = 5; p = (n - 1)^2; A0 = ones(n); maxdet = 0;
for k = 0 : 2^p - 1
    Asub = rem(floor(k * 2.^(n - 1 : 0)), 2); A = A0; A(2 : n, 2 : n) = 1 - 2*Asub
    reshape(Asub, n - 1, n - 1);
    if abs(det(A)) > maxdet, maxdet = abs(det(A)); maxA = A; end
end
```

Output: $\text{maxA} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{bmatrix}$ $\text{maxdet} = 48$. 


Reduce $B$ by row operations to $[\text{row 3; row 2; row 1}]$. Then $\det B = -6$ (odd permutation from the order of the rows in $A$).

Problem Set 5.2, page 266

1 $\det A = 1 + 18 + 12 - 9 - 4 - 6 = 12$, the rows of $A$ are independent; $\det B = 0$, row 1 + row 2 = row 3 so the rows of $B$ are linearly dependent; $\det C = -1$, so $C$ has independent rows ($\det C$ has one term, an odd permutation).

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = -1$, independent but $\det D = 0$ because its submatrix $B$ has dependent rows.

3 The problem suggests 3 ways to see that $\det A = 0$: All cofactors of row 1 are zero. $A$ has rank $\leq 2$. Each of the 6 terms in $\det A$ is zero. Notice also that column 2 has no pivot.

4 $a_{11}a_{23}a_{32}a_{44}$ gives $-1$, because the terms $a_{23}a_{32}$ have columns 2 and 3 in reverse order. $a_{14}a_{23}a_{32}a_{41}$ which has 2 row exchanges gives $+1$, $\det A = 1 - 1 = 0$. Using the same entries but now taken from $B$, $\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 64 - 16 = 48$.

5 Four zeros in the same row guarantee $\det = 0$ (and also four zeros in the same column). $A = I$ has 12 zeros (this is the maximum with $\det \neq 0$).

6 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms will be zeros (b) 15 terms must be zero. Effectively we are counting the permutations that make everyone move; 2, 3, 1 and 3, 1, 2 for $n = 3$ mean that the other 4 permutations take a term from the diagonal of $A$; so those terms are 0 when the diagonal is all zeros.

7 $5!/2 = 60$ permutation matrices (half of $5! = 120$ permutations) have $\det = +1$. Move row 5 of $I$ to the top; then starting from $(5, 1, 2, 3, 4)$ elimination will do four row exchanges on $P$.

8 If $\det A \neq 0$, then certainly some term $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ in the big formula is not zero! Move rows 1, 2, \ldots, $n$ into rows $\alpha, \beta, \ldots, \omega$. Then all these nonzero $a$’s will be on the main diagonal.
9 The big formula has six terms all $\pm 1$: say $q$ are $-1$ and $6 - q$ are $1$. Then $\det A = -q + 6 - q = \text{even}$ (so $\det A = 5$ is impossible). Also $\det A = 6$ is impossible. All $3$ even permutations like $a_{11}a_{22}a_{33}$ would have to give $+1$ (so an even number of $-1$'s in the matrix). But all $3$ odd permutations like $a_{11}a_{23}a_{32}$ would have to give $-1$ (so an odd number of $-1$'s in the matrix). We can’t have it both ways, so $\det A = 4$ is best possible and not hard to arrange: put $-1$'s on the main diagonal.

10 The $4!/2 = 12$ even permutations are $(1, 2, 3, 4), (2, 1, 4, 3), (3, 1, 4, 2), (4, 3, 2, 1)$, and $8$ $P$'s with one number in place and even permutation of the other three numbers: examples are $1, 3, 4, 2$ and $1, 4, 2, 3$.

$$\det(I + P_{\text{even}})$$ is always $16$ or $4$ or $0$ ($16$ comes from $I + I$).

11 $C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ and $AC^T = (ad - bc)I$ and $D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$.

$$\det B = 1(0) + 2(42) + 3(-35) = -21.$$

12 $A^{-1} = C^T / \det A = C^T / 4$.

13 (a) $C_1 = 0$, $C_2 = -1$, $C_3 = 0$, $C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row $1$ then cofactors of column $1$. Therefore $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$.

14 For the matrices in Problem 13 to produce nonzeros in the big formula, we must choose $1$'s from column $2$ then column $1$, column $4$ then column $3$, and so on. Therefore $n$ must be even to have $\det \neq 0$. The number of row exchanges is $n/2$ so the overall determinant is $C_n = (-1)^{n/2}$.

15 The $1, 1$ cofactor of the $n$ by $n$ matrix is $E_{n-1}$. The $1, 2$ cofactor has a single $1$ in its first column, with cofactor $E_{n-2}$: sign gives $-E_{n-2}$. So $E_n = E_{n-1} - E_{n-2}$. Then $E_1$ to $E_6$ is $1, 0, -1, -1, 0, 1$ and this cycle of six will repeat: $E_{100} = E_4 = -1$.

16 The $1, 1$ cofactor of the $n$ by $n$ matrix is $F_{n-1}$. The $1, 2$ cofactor has a $1$ in column $1$, with cofactor $F_{n-2}$. Multiply by $(-1)^{1+2}$ and also $(-1)$ from the $1, 2$ entry to find $F_n = F_{n-1} + F_{n-2}$. So these determinants are Fibonacci numbers.
17 Use cofactors along row 4 instead of row 1 (last row instead of first).

\[ |B_4| = 2 \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ -1 & 2 \end{bmatrix} + \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix} = 2|B_3| - \det \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}. \]

So \(|B_4| = 2|B_3| - |B_2|\).

18 Rule 3 (linearity in row 1) gives \(|B_n| = |A_n| - |A_{n-1}| = (n+1) - n = 1.\)

19 Since \(x, x^2, x^3\) are all in the same row, they never multiply each other in \(\det V_4\).

The determinant is zero at \(x = a\) or \(b\) or \(c\) because of equal rows! So \(\det V\) has factors 
\( (x-a)(x-b)(x-c) \). Multiply by the cofactor \(V_3\).

The Vandermonde matrix \(V_{ij} = (x_i)^{j-1}\) is for fitting a polynomial \(p(x) = b\) at the points \(x_i\).

It has \(\det V = \) product of all \(x_k - x_m\) for \(k > m\).

20 \(G_2 = -1, G_3 = 2, G_4 = -3,\) and \(G_n = (-1)^{n-1}(n-1)\). One way to reach that \(G_n\) is to multiply the \(n\) eigenvalues \(-1, -1, \ldots, -1, n - 1\) of the matrix. Is there a good choice of row operations to produce this determinant \(G_n\) ?

21 \(S_1 = 3, S_2 = 8, S_3 = 21.\) The rule looks like every second number in Fibonacci’s sequence \(\ldots 3, 5, 8, 13, 21, 34, 55, \ldots\) so the guess is \(S_4 = 55.\)

Following the solution to Problem 30 with 3’s instead of 2’s on the diagonal confirms \(S_4 = 81 + 1 - 9 - 9 - 9 = 55.\)

Problem 32 directly proves \(S_n = F_{2n+2}.\)

22 Changing 3 to 2 in the corner reduces the determinant \(F_{2n+2}\) by 1 times the cofactor of that corner entry. This cofactor is the determinant of \(S_{n-1}\) (one size smaller) which is \(F_{2n}\).

Therefore changing 3 to 2 changes the determinant to \(F_{2n+2} - F_{2n},\) which is Fibonacci’s \(F_{2n+1}.\)

23 (a) If we choose an entry from \(B\) we must choose an entry from the zero block; result zero. This leaves entries from \(A\) times entries from \(D\) leading to \((\det A)(\det D)\)

(b) and (c) Take \(A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.\) See #25.

24 (a) All the lower triangular blocks \(L_k\) have 1’s on the diagonal and \(\det = 1.\)

Then use \(A_k = L_kU_k\) to find \(\det U_k = \det A_k = 2, 6, -6\) for \(k = 1, 2, 3.\)
Solutions to Exercises

(b) Equation (3) in this section gives the $k$th pivot as $\det A_k / \det A_{k-1}$. So $\det A_k = 5, 6, 7$ gives pivot $d_k = 5/1, 6/5, 7/6$.

25 Problem 23 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$. By the product rule this is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

26 If $A$ is a row and $B$ is a column then $\det M = \det AB = \det$ product of $A$ and $B$. If $A$ is a column and $B$ is a row then $AB$ has rank 1 and $\det M = \det AB = 0$ (unless $m = n = 1$). This block matrix $M$ is invertible when $AB$ is invertible which certainly requires $m \leq n$.

27 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. Derivative with respect to $a_{11} = \text{cofactor } C_{11}$.

28 Row 1 $- 2$ row 2 $+ 2$ row 3 $= 0$ so this matrix is singular and $\det A = 0$.

29 There are five nonzero products, all 1’s with a plus or minus sign. Here are the (row, column) numbers and the signs: $+ (1, 1)(2, 2)(3, 3)(4, 4) + (1, 2)(2, 1)(3, 4)(4, 3) - (1, 2)(2, 1)(3, 3)(4, 4) - (1, 1)(2, 2)(3, 4)(4, 3) - (1, 1)(2, 3)(3, 2)(4, 4)$. Total $-1$.

30 The 5 products in solution 29 change to $16 + 1 - 4 - 4 - 4$ since $A$ has 2’s and 1’s:

$$(2)(2)(2)(2) + (-1)(-1)(-1)(-1)(-1)(2)(2)(2)(-1)(-1)(-1) - (2)(2)(-1)(-1) - (2)(-1)(-1)(2) = 5 = n + 1.$$  

31 $\det P = -1$ because the cofactor of $P_{14} = 1$ in row one has sign $(-1)^{1+4}$. The big formula for $\det P$ has only one term $(1 \cdot 1 \cdot 1 \cdot 1)$ with minus sign because three exchanges take 4, 1, 2, 3 into 1, 2, 3, 4; $\det(P^2) = (\det P)(\det P) = +1$ so

$\det \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not right.

32 The problem is to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci’s rule:

$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}$. 

33 The difference from 20 to 19 multiplies its 3 by 3 cofactor = 1: then det drops by 1.

34 (a) The last three rows must be dependent because only 2 columns are nonzero

(b) In each of the 120 terms: Choices from the last 3 rows must use 3 different columns; at least one of those choices will be zero.

35 Subtracting 1 from the n, n entry subtracts its cofactor \( C_{nn} \) from the determinant. That cofactor is \( C_{nn} = 1 \) (smaller Pascal matrix). Subtracting 1 from 1 leaves 0.

**Problem Set 5.3, page 283**

1 (a) \( |A| = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3, \ |B_1| = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 6, \ |B_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 \) so \( x_1 = \frac{-6}{3} = -2 \) and \( x_2 = \frac{3}{3} = 1 \) (b) \( |A| = 4, |B_1| = 3, |B_2| = 2, |B_3| = 1. \) Therefore \( x_1 = \frac{3}{4} \) and \( x_2 = -\frac{1}{2} \) and \( x_3 = \frac{1}{4}. \)

2 (a) \( y = \frac{a}{c} \left| \begin{array}{cc} 1 & b \\ 0 & c \end{array} \right| = \frac{a}{c} (ad - bc) \) (b) \( y = \det B_2/\det A = (fg - id)/D. \)

That is because \( B_2 \) with \( (1, 0, 0) \) in column 2 has \( \det B_2 = fg - id. \)

3 (a) \( x_1 = 3/0 \) and \( x_2 = -2/0: no \ solution \) (b) \( x_1 = x_2 = 0/0: undetermined. \)

4 (a) \( x_1 = \det([b \ a_2 \ a_3])/\det A, \) if \( \det A \neq 0. \) This is \( |B_1|/|A|. \)

(b) The determinant is linear in its first column so \( |a_1 a_2 a_3| + x_2|a_2 a_3 a_3| + x_3|a_3 a_2 a_3| \) splits into \( x_1|a_1 a_2 a_3| + x_2|a_2 a_3 a_3| + x_3|a_3 a_2 a_3|. \) The last two determinants are zero because of repeated columns, leaving \( x_1|a_1 a_2 a_3| \) which is \( x_1 \det A. \)

5 If the first column in \( A \) is also the right side \( b \) then \( \det A = \det B_1. \) Both \( B_2 \) and \( B_3 \) are singular since a column is repeated. Therefore \( x_1 = |B_1|/|A| = 1 \) and \( x_2 = x_3 = 0. \)

6 (a) \( \begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \) (b) \( \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \)

An invertible symmetric matrix has a symmetric inverse.
7 If all cofactors = 0 then $A^{-1}$ would be the zero matrix if it existed; cannot exist. (And also, the cofactor formula gives $\det A = 0$.) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has no zero cofactors but it is not invertible.

8 $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. This is $(\det A)I$ and $(\det A) = 3$.

9 If we know the cofactors and $(\det A)^n$, the inverse of $A$ can be found from the cofactor matrix for $C$.

10 Take the determinant of $AC^T = (\det A)I$. The left side gives $\det AC^T = (\det A)(\det C)$ while the right gives $(\det A)^n$. Divide by $\det A$ to reach $C = (\det A)^{n-1}$.

11 The cofactors of $A$ are integers. Division by $\pm 1$ gives integer entries in $A^{-1}$.

12 Both $\det A$ and $(\det A)^{-1}$ are integers since the matrices contain only integers. But $\det A^{-1} = 1/\det A$ so $\det A$ must be 1 or -1.

13 $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ has cofactor matrix $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{bmatrix}$ and $A^{-1} = \frac{1}{5}C^T$.

14 (a) Lower triangular $L$ has cofactors $C_{21} = C_{31} = C_{32} = 0$ (b) $C_{12} = C_{21}$,

15 For $n = 5$, $C$ contains 25 cofactors and each 4 by 4 cofactor has 24 terms. Each term needs 3 multiplications: total 1800 multiplications vs. 125 for Gauss-Jordan.

16 (a) Area $\frac{1}{2} \begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix} = 10$ (b) and (c) Area $10/2 = 5$, these triangles are half of the parallelogram in (a).

17 Volume $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} = 20$. Area of faces = $\begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2i - 2j + 8k$

18 (a) Area $\frac{1}{2} \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 1 \\ 5 & 1 & 1 \end{bmatrix} = 5$ (b) new triangle area $\frac{1}{2} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ 1 & 0 & 1 \end{bmatrix} = 5 + 7 = 12$.

19 $\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$ because the transpose has the same determinant. See #22.
The edges of the hypercube have length $\sqrt{1 + 1 + 1 + 1} = 2$. The volume $\det H$ is $2^4 = 16$. ($H/2$ has orthonormal columns. Then $\det(H/2) = 1$ leads again to $\det H = 16$ in 4 dimensions.)

The maximum volume $L_1L_2L_3L_4$ is reached when the edges are orthogonal in $\mathbb{R}^4$. With entries 1 and $-1$ all lengths are $\sqrt{4} = 2$. The maximum determinant is $2^4 = 16$, achieved in Problem 20. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can’t be achieved by $\pm 1$. $\rho^2 \sin \phi d\rho d\phi d\theta$.

This question is still waiting for a solution! An 18.06 student showed me how to transform the parallelogram for $A$ to the parallelogram for $A^T$, without changing its area. (Edges slide along themselves, so no change in baselength or height or area.)

$$A^T A = \begin{bmatrix} a^T & b^T & c^T \end{bmatrix} \begin{bmatrix} a^T a & 0 & 0 \\ 0 & b^T b & 0 \\ 0 & 0 & c^T c \end{bmatrix}$$ has $\det A^T A = (\|a\|\|b\|\|c\|)^2$ and $\det A = \pm \|a\|\|b\|\|c\|$.

The box has height 4 and volume $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = 4. i \times j = k$ and $(k \cdot w) = 4$.

The $n$-dimensional cube has $2^n$ corners, $n2^{n-1}$ edges and $2n (n-1)$-dimensional faces. Coefficients from $(2+x)^n$ in Worked Example 2.4A. Cube from 2I has volume $2^n$.

The pyramid has volume $\frac{1}{6}$. The 4-dimensional pyramid has volume $\frac{1}{24}$ (and $\frac{1}{n!}$ in $\mathbb{R}^n$).

$x = r \cos \theta, y = r \sin \theta$ give $J = r$. This is the $r$ in polar area $r \, dr \, d\theta$. The columns are orthogonal and their lengths are 1 and $r$.

$$J = \begin{bmatrix} \sin \phi \cos \theta & \rho \cos \phi \sin \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & \theta \end{bmatrix} = \rho^2 \sin \phi. \text{ This Jacobian is needed}$$

for triple integrals inside spheres. Those integrals have $\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$. 

Solutions to Exercises

29 From \( x, y \) to \( r, \theta \):
\[
\begin{pmatrix}
\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}
\end{pmatrix}
= \begin{pmatrix}
x/r & y/r
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & \sin \theta
\end{pmatrix}
\begin{pmatrix}
-x/r^2 & x/r^2
\end{pmatrix}
= \begin{pmatrix}
-\sin \theta/r & (\cos \theta)/r
\end{pmatrix}
\]
\[
= \frac{1}{r} \text{ Jacobian in 27.}
\]
The surprise was that \( \frac{dx}{dr} \) and \( \frac{dy}{dr} \) are both \( \frac{1}{r} \).

30 The triangle with corners \((0,0), (6,0), (1,4)\) has area \( (6)(4)/2 = 12 \). Rotated by \( \theta = 60^\circ \) the area is unchanged. The determinant of the rotation matrix is
\[
J = \begin{vmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{vmatrix} = \begin{vmatrix}
1/2 & -\sqrt{3}/2 \\
\sqrt{3}/2 & 1/2
\end{vmatrix} = 1.
\]

31 Base area \( ||u \times v|| = 10 \), height \( ||w|| \cos \theta = 2 \), volume \((10)(2) = 20\).

32 The volume of the box is \( \det \begin{vmatrix}
2 & 4 & 0 \\
-1 & 3 & 0 \\
1 & 2 & 2
\end{vmatrix} = 20 \), agreeing with Problem 31.

33 \( \begin{vmatrix}
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3 \\
w_1 & w_2 & w_3
\end{vmatrix} = u_1 \begin{vmatrix}
v_2 & v_3 \\
w_2 & w_3
\end{vmatrix} - u_2 \begin{vmatrix}
v_1 & v_3 \\
w_1 & w_3
\end{vmatrix} + u_3 \begin{vmatrix}
v_1 & v_2 \\
w_1 & w_2
\end{vmatrix} \). This is \( u \cdot (v \times w) \).

34 \( (w \times u) \cdot v = (v \times w) \cdot u = (u \times v) \cdot w \): Even permutation of \( u, v, w \) keeps the same determinant. Odd permutations like \( (u \times v) \cdot v \) will reverse the sign.

35 \( S = (2,1,-1) \), area \( ||PQ \times PS|| = ||(-2,-2,-1)|| = \sqrt{2^2 + 2^2 + 1^2} = 3 \). The other four corners of the box can be \((0,0,0), (0,0,2), (1,2,2), (1,1,0)\). The volume of the tilted box with edges along \( P, Q, \) and \( R \) is \( |\det| = 1 \).

36 If \((1,1,0), (1,2,1), (x,y,z)\) are in a plane the volume is \( \det \begin{vmatrix}
x & y & z \\
1 & 1 & 0 \\
1 & 2 & 1
\end{vmatrix} = x - y + z = 0 \).

The “box” with those edges is flattened to zero height.

37 \( \det \begin{vmatrix}
x & y & z \\
2 & 3 & 1 \\
1 & 2 & 3
\end{vmatrix} = 7x - 5y + z \) will be zero when \((x,y,z)\) is a combination of \((2,3,1)\) and \((1,2,3)\). The plane containing those two vectors has equation \( 7x - 5y + z = 0 \).

Volume = zero because the 3 box edges out from \((0,0,0)\) lie in a plane.
Doubling each row multiplies the volume by $2^n$. Then $2 \det A = \det(2A)$ only if $n = 1$.

$AC^T = (\det A)I$ gives $(\det A)(\det C) = (\det A)^n$. Then $\det A = (\det C)^{1/n}$ with $n = 4$. With $\det A^{-1} = 1/\det A$, construct $A^{-1}$ using the cofactors. Invert to find $A$.

The cofactor formula adds 1 by 1 determinants (which are just entries) times their cofactors of size $n - 1$. Jacobi discovered that this formula can be generalized. For $n = 5$, Jacobi multiplied each 2 by 2 determinant from rows 1-2 (with columns $a < b$) times a 3 by 3 determinant from rows 3-5 (using the remaining columns $c < d < e$).

The key question is $+$ or $-$ sign (as for cofactors). The product is given a $+$ sign when $a, b, c, d, e$ is an even permutation of $1, 2, 3, 4, 5$. This gives the correct determinant $+1$ for that permutation matrix. More than that, all other $P$ that permute $a, b$ and separately $c, d, e$ will come out with the correct sign when the 2 by 2 determinant for columns $a, b$ multiplies the 3 by 3 determinant for columns $c, d, e$.

The Cauchy-Binet formula gives the determinant of a square matrix $AB$ (and $AA^T$ in particular) when the factors $A, B$ are rectangular. For $(2 \times 3)$ times $(3 \times 2)$ there are 3 products of 2 by 2 determinants from $A$ and $B$ (printed in boldface):

$$
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
\end{bmatrix}
\begin{bmatrix}
  g & j \\
  h & k \\
  i & \ell \\
\end{bmatrix}
= 
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
\end{bmatrix}
\begin{bmatrix}
  g & j \\
  h & k \\
  i & \ell \\
\end{bmatrix}
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
\end{bmatrix}
\begin{bmatrix}
  g & j \\
  h & k \\
  i & \ell \\
\end{bmatrix}
$$

Check $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}$, $AB = \begin{bmatrix} 14 & 30 \\ 30 & 66 \end{bmatrix}$

Cauchy-Binet: $(4 - 2)(4 - 2) + (7 - 3)(7 - 3) + (14 - 12)(14 - 12) = 24$

det of $AB$ : $(14)(66) - (30)(30) = 24$

A 5 by 5 tridiagonal matrix has cofactor $C_{11} = 4$ by 4 tridiagonal matrix. Cofactor $C_{12}$ has only one nonzero at the top of column 1. That nonzero multiplies its 3 by 3 cofactor which is tridiagonal. So $\det A = a_{11}C_{11} + a_{12}C_{12}$ = tridiagonal determinants of sizes 4 and 3. The number $F_n$ of nonzero terms in $\det A$ follows Fibonacci’s rule $F_n = F_{n-1} + F_{n-2}$.