INTRODUCTION

TO

LINEAR

ALGEBRA

Fifth Edition

MANUAL FOR INSTRUCTORS

Gilbert Strang

Massachusetts Institute of Technology

math.mit.edu/linearalgebra

web.mit.edu/18.06

video lectures: ocw.mit.edu

math.mit.edu/∼gs

www.wellesleycambridge.com
email: linearalgebrabook@gmail.com

Wellesley - Cambridge Press

Box 812060

Wellesley, Massachusetts 02482

Problem Set 3.1, page 131

Note An interesting "max-plus" vector space comes from the real numbers $\mathbf{R}$ combined with $-\infty$. Change addition to give $x + y = \mathbf{max}(\boldsymbol{x}, \boldsymbol{y})$ and change multiplication to $xy =$ **usual** $\boldsymbol{x} + \boldsymbol{y}$. Which y is the zero vector that gives $x + \mathbf{0} = \max(x, \mathbf{0}) = x$ for every x?

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

2 When $c(x_1, x_2) = (cx_1, 0)$, the only broken rule is 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$. Rules (1)-(4) for addition $\boldsymbol{x} + \boldsymbol{y}$ still hold since addition is not changed.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$
(b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ this is $3(\mathbf{2} + \mathbf{1}) = 8$. The zero vector is the number 1.

4 The zero vector in matrix space $\mathbf{M}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$.
The smallest subspace of $\mathbf{M}$ containing the matrix A consists of all matrices cA.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) Matrices whose main diagonal is all zero.

6 When $\boldsymbol{f}(x) = x^2$ and $\boldsymbol{g}(x) = 5x$, the combination $3\boldsymbol{f} - 4\boldsymbol{g}$ in function space is $\boldsymbol{h}(x) = 3\boldsymbol{f}(x) - 4\boldsymbol{g}(x) = 3x^2 - 20x$.

7 Rule 8 is broken: If $c\boldsymbol{f}(x)$ is defined to be the usual $\boldsymbol{f}(cx)$ then $(c_1 + c_2)\boldsymbol{f} = \boldsymbol{f}((c_1 + c_2)x)$ is not generally the same as $c_1\boldsymbol{f} + c_2\boldsymbol{f} = \boldsymbol{f}(c_1x) + \boldsymbol{f}(c_2x)$.

8 If $(\boldsymbol{f} + \boldsymbol{g})(x)$ is the usual $\boldsymbol{f}(\boldsymbol{g}(x))$ then $(\boldsymbol{g} + \boldsymbol{f})x$ is $\boldsymbol{g}(\boldsymbol{f}(x))$ which is different. In Rule 2 both sides are $\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{h}(x)))$. Rule 4 is broken because there might be no inverse function $\boldsymbol{f}^{-1}(x)$ such that $\boldsymbol{f}(\boldsymbol{f}^{-1}(x)) = x$. If the inverse function exists it will be the vector $-\boldsymbol{f}$.

9 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$
(b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions: $(1, 1) + (-1, 1) = (0, 2)$ is removed.

10 The only subspaces are (a) the plane with $b_1 = b_2$ (d) the linear combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ (e) the plane with $b_1 + b_2 + b_3 = 0$.

11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

12 For the plane $x + y - 2z = 4$, the sum of $(4, 0, 0)$ and $(0, 4, 0)$ is not on the plane. (The key is that this plane does not go through $(0, 0, 0)$.)

13 The parallel plane $\mathbf{P}_0$ has the equation $x + y - 2z = 0$. Pick two points, for example $(2, 0, 1)$ and $(0, 2, 1)$, and their sum $(2, 2, 2)$ is in $\mathbf{P}_0$.

14 (a) The subspaces of $\mathbf{R}^2$ are $\mathbf{R}^2$ itself, lines through $(0, 0)$, and $(0, 0)$ by itself (b) The subspaces of $\mathbf{D}$ are $\mathbf{D}$ itself, the zero matrix by itself, and all the "one-dimensional" subspaces that contain all multiples of one fixed matrix :

$$c \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \quad \text{for all } c.$$

15 (a) Two planes through $(0, 0, 0)$ probably intersect in a line through $(0, 0, 0)$

(b) The plane and line probably intersect in the point $(0, 0, 0)$. *Could be a line* !

(c) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are in both $\boldsymbol{S}$ and $\boldsymbol{T}$, $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are in both subspaces.

16 The smallest subspace containing a plane $\mathbf{P}$ and a line $\mathbf{L}$ is *either* $\mathbf{P}$ (when the line $\mathbf{L}$ is in the plane $\mathbf{P}$) *or* $\mathbf{R}^3$ (when $\mathbf{L}$ is not in $\mathbf{P}$).

17 (a) The invertible matrices do not include the zero matrix, so they are not a subspace

(b) The sum of singular matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not singular: not a subspace.

18 (a) *True*: The symmetric matrices do form a subspace (b) *True*: The matrices with $A^{\mathrm{T}} = -A$ do form a subspace (c) *False*: The sum of two unsymmetric matrices could be symmetric.

19 The column space of A is the x-axis = all vectors $(x, 0, 0)$: a *line*. The column space of B is the xy plane = all vectors $(x, y, 0)$. The column space of C is the line of vectors $(x, 2x, 0)$.

20 (a) Elimination leads to $0 = b_2 - 2b_1$ and $0 = b_1 + b_3$ in equations 2 and 3: Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Elimination leads to $0 = b_1 + b_3$ in equation 3: Solution only if $b_3 = -b_1$.

21 A combination of the columns of C is also a combination of the columns of A. Then $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ have the same column space. $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has a different column space. The key word is "space".

22 (a) Solution for every $\boldsymbol{b}$ (b) Solvable only if $b_3 = 0$ (c) Solvable only if $b_3 = b_2$.

23 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is ***already in*** the column space. $[\,A \;\; \boldsymbol{b}\,] = \begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} \end{bmatrix}$ (larger column space) (no solution to $A\boldsymbol{x} = \boldsymbol{b}$) $\begin{bmatrix} 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ ($\boldsymbol{b}$ is in column space) ($A\boldsymbol{x} = \boldsymbol{b}$ has a solution)

24 The column space of AB is ***contained in*** (possibly equal to) the column space of A. The example $B =$ zero matrix and $A \neq 0$ is a case when $AB =$ zero matrix has a smaller column space (it is just the zero space $\mathbf{Z}$) than A.

25 The solution to $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in $\boldsymbol{C}(A)$ so is $\boldsymbol{b} + \boldsymbol{b}^*$.

26 The column space of any invertible 5 by 5 matrix is $\mathbf{R}^5$. The equation $A\boldsymbol{x} = \boldsymbol{b}$ is always solvable (by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$) so every $\boldsymbol{b}$ is in the column space of that invertible matrix.

27 (a) ***False***: Vectors that are ***not*** in a column space don't form a subspace. (b) ***True***: Only the zero matrix has $\boldsymbol{C}(A) = \{\mathbf{0}\}$. (c) ***True***: $\boldsymbol{C}(A) = \boldsymbol{C}(2A)$. (d) ***False***: $\boldsymbol{C}(A - I) \neq \boldsymbol{C}(A)$ when $A = I$ or $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (or other examples).

28 $A = \begin{bmatrix} 1 & 1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ 0 & 1 & \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & \mathbf{2} \\ 1 & 0 & \mathbf{1} \\ 0 & 1 & \mathbf{1} \end{bmatrix}$ do not have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\boldsymbol{C}(A)$. $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ has $\boldsymbol{C}(A) =$ line in $\mathbf{R}^3$.

29 When $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for all $\boldsymbol{b}$, every $\boldsymbol{b}$ is in the column space of A. So that space is $\boldsymbol{C}(A) = \mathbf{R}^9$.

30 (a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\boldsymbol{S}+\boldsymbol{T}$, then $\boldsymbol{u}=\boldsymbol{s}_1+\boldsymbol{t}_1$ and $\boldsymbol{v}=\boldsymbol{s}_2+\boldsymbol{t}_2$. So $\boldsymbol{u}+\boldsymbol{v}=(\boldsymbol{s}_1+\boldsymbol{s}_2)+(\boldsymbol{t}_1+\boldsymbol{t}_2)$ is also in $\boldsymbol{S}+\boldsymbol{T}$. And so is $c\boldsymbol{u}=c\boldsymbol{s}_1+c\boldsymbol{t}_1$: $\boldsymbol{S}+\boldsymbol{T}=$ *subspace*.

(b) If $\boldsymbol{S}$ and $\boldsymbol{T}$ are different lines, then $\boldsymbol{S}\cup\boldsymbol{T}$ is just the two lines (*not a subspace*) but $\boldsymbol{S}+\boldsymbol{T}$ is the whole plane that they span.

31 If $\boldsymbol{S}=\boldsymbol{C}(A)$ and $\boldsymbol{T}=\boldsymbol{C}(B)$ then $\boldsymbol{S}+\boldsymbol{T}$ is the column space of $M=[\,A\quad B\,]$.

32 The columns of AB are combinations of the columns of A. So all columns of $[\,A\quad AB\,]$ are already in $\boldsymbol{C}(A)$. But $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ has a larger column space than $A^2=\begin{bmatrix}0&0\\0&0\end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ exactly when A is *invertible*.

Problem Set 3.2, page 142

1 (a) $U=\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$ Free variables x_2,x_4,x_5 Pivot variables x_1,x_3 (b) $U=\begin{bmatrix}2&4&2\\0&4&4\\0&0&0\end{bmatrix}$ Free x_3 Pivot x_1,x_2

2 (a) Free variables x_2,x_4,x_5 and solutions $(-2,1,0,0,0)$, $(0,0,-2,1,0)$, $(0,0,-3,0,1)$

(b) Free variable x_3: solution $(1,-1,1)$. Special solution for each free variable.

3 $R=\begin{bmatrix}1&2&0&0&0\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$, $R=\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}$, R has the same nullspace as U and A.

4 (a) Special solutions $(3,1,0)$ and $(5,0,1)$ (b) $(3,1,0)$. **Total of pivot and free is $\boldsymbol{n}$.**

5 (a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has *no* free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

6 $\begin{bmatrix}0&\mathbf{1}&1&1&1&1&1\\0&0&0&\mathbf{1}&1&1&1\\0&0&0&0&\mathbf{1}&1&1\\0&0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}\mathbf{1}&1&1&1&1&1&1\\0&0&\mathbf{1}&1&1&1&1\\0&0&0&0&0&\mathbf{1}&1\\0&0&0&0&0&0&\mathbf{1}\end{bmatrix}\begin{bmatrix}0&0&0&\mathbf{1}&1&1&1\\0&0&0&0&0&\mathbf{1}&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}$

7 $\begin{bmatrix} \mathbf{1} & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{1} & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Notice the identity matrix in the pivot columns of these ***reduced*** row echelon forms R.

8 If column 4 of a 3 by 5 matrix is all zero then x_4 is a ***free*** variable. Its special solution is $\boldsymbol{x} = (0, 0, 0, 1, 0)$, because 1 will multiply that zero column to give $A\boldsymbol{x} = \mathbf{0}$.

9 If column $1 =$ column 5 then x_5 is a free variable. Its special solution is $(-1, 0, 0, 0, 1)$.

10 If a matrix has n columns and r pivots, there are $\boldsymbol{n} - \boldsymbol{r}$ special solutions. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $\boldsymbol{r} = \boldsymbol{n}$. The column space is all of $\mathbf{R}^m$ when $r = m$. All those statements are important!

11 The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

12 $A = [1 \;\; -3 \;\; -1]$ gives the plane $x - 3y - z = 0$; y and z are free variables. The special solutions are $(3, 1, 0)$ and $(1, 0, 1)$.

13 Fill in $\mathbf{12}$ then $\mathbf{3}$ then $\mathbf{1}$ to get the complete solution in $\mathbf{R}^3$ to $x - 3y - z = 12$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \mathbf{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \mathbf{1} \\ 0 \\ 1 \end{bmatrix}$ = one particular solution + all nullspace solutions.

14 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1, 0, 1, 0, 1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (this nullspace is a line in $\mathbf{R}^5$).

15 To produce special solutions $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$ with free variables x_3, x_4: $R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$ and A can be any invertible 2 by 2 matrix times this R.

16 The nullspace of $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ is the line through the special solution $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

17 $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in $\boldsymbol{N}(A)$. ***Which other*** A's?

18 This construction is impossible for 3 by 3 ! 2 pivot columns and 2 free variables.

19 $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ has $(1, 1, 1)$ in $\boldsymbol{C}(A)$ and only the line (c, c, c, c) in $\boldsymbol{N}(A)$.

20 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$. Notice that rref$(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not A^{T}.

21 If nullspace $=$ column space (with r pivots) then $n - r = r$. If $n = 3$ then $3 = 2r$ is impossible.

22 If A times every column of B is zero, the column space of B is contained in the ***nullspace*** of A. An example is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Here $\boldsymbol{C}(B)$ ***equals*** $\boldsymbol{N}(A)$. For $B = 0, \boldsymbol{C}(B)$ is smaller than $\boldsymbol{N}(A)$.

23 For $A =$ random 3 by 3 matrix, R is almost sure to be I. For 4 by 3, R is most likely to be I with a fourth row of zeros. What is R for a random 3 by 4 matrix?

24 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ shows that (a)(b)(c) are all false. Notice rref$(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

25 If $\boldsymbol{N}(A) =$ line through $\boldsymbol{x} = (2, 1, 0, 1)$, A has ***three pivots*** (4 columns and 1 special solution). Its reduced echelon form can be $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).

26 $R = [\,1 \;\; -2 \;\; -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$. Any zero rows come after those rows.

27 (a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}$ (b) All 8 matrices are R's !

28 One reason that R is the same for A and $-A$: They have the same nullspace. (They also have the same row space. They also have the same column space, but that is not required for two matrices to share the same R. R tells us the nullspace and row space.)

29 The nullspace of $B = [\,A \quad A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix}\boldsymbol{y}\\-\boldsymbol{y}\end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

30 If $C\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection*.

31 (a) $R = \begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ rank 1 (b) $R = \begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\end{bmatrix}$ rank 2

(c) $R = \begin{bmatrix}1&-1&1&-1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ rank 1

32 $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops. The solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ are combinations of $(-1, 0, 0, 1, -1, 0)$ and $(0, 0, -1, -1, 0, 1)$ and $(0, -1, 0, 0, 1, -1)$. Those are flows around the 3 small loops.

33 (a) and (c) are correct; (b) is completely false; (d) is false because R might have 1's in nonpivot columns.

34 $R_A = \begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}$ $R_B = \begin{bmatrix}R_A & R_A\end{bmatrix}$ $R_C \longrightarrow \begin{bmatrix}R_A&0\\0&R_A\end{bmatrix} \longrightarrow$ Zero rows go to the bottom

35 If all pivot variables come last then $R = \begin{bmatrix}0&I\\0&0\end{bmatrix}$. The nullspace matrix is $N = \begin{bmatrix}I\\0\end{bmatrix}$.

36 I think $R_1 = A_1, R_2 = A_2$ is true. But $R_1 - R_2$ may have -1's in some pivots.

37 A and A^{T} have the same rank r = number of pivots. But *pivcol* (the column number) is 2 for this matrix A and 1 for A^{T}: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

38 Special solutions in $N = [\,-2 \;\; -4 \;\; 1 \;\; 0;\; -3 \;\; -5 \;\; 0 \;\; 1\,]$ and $[\,1 \;\; 0 \;\; 0;\; 0 \;\; -2 \;\; 1\,]$.

39 The new entries keep rank 1: $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & \mathbf{4} & \mathbf{8} \\ 4 & \mathbf{8} & \mathbf{16} \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{2} & 6 & -\mathbf{3} \\ 1 & \mathbf{3} & -\mathbf{3/2} \\ 2 & 6 & -3 \end{bmatrix}$, $M = \begin{bmatrix} a & b \\ c & \boldsymbol{bc/a} \end{bmatrix}$.

40 If A has rank 1, the column space is a ***line*** in $\mathbf{R}^m$. The nullspace is a ***plane*** in $\mathbf{R}^n$ (given by one equation). The nullspace matrix N is n by $n-1$ (with $n-1$ special solutions in its columns). The column space of A^{T} is a ***line*** in $\mathbf{R}^n$.

41 $\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}$

42 With rank 1, the second row of R is a zero row.

43 Invertible r by r submatrices
Use pivot rows and columns $S = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $S = [\,1\,]$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

44 P has $\operatorname{rank} r$ (the same as A) because elimination produces the same pivot columns.

45 The rank of R^{T} is also r. The example matrix A has $\operatorname{rank} 2$ with invertible S:

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix} \qquad P^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix} \qquad S^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

46 The product of rank one matrices has rank one or zero. These particular matrices have rank$(AB) = 1$; rank$(AC) = 1$ except $AC = 0$ if $c = -1/2$.

47 $(\boldsymbol{uv}^{\mathrm{T}})(\boldsymbol{wz}^{\mathrm{T}}) = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w})\boldsymbol{z}^{\mathrm{T}}$ has rank one unless the inner product is $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.

48 (a) By matrix multiplication, each column of AB is A times the corresponding column of B. So if column j of B is a combination of earlier columns, then column j of AB is the same combination of earlier columns of AB. Then rank $(AB) \leq$ rank (B). No new pivot columns! (b) The rank of B is $r = 1$. Multiplying by A cannot increase this rank. The rank of AB stays the same for $A_1 = I$ and $B = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$. It drops to zero for $A_2 = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{-1} \end{bmatrix}$.

49 If we know that $\operatorname{rank}(B^{\mathrm{T}}A^{\mathrm{T}}) \leq \operatorname{rank}(A^{\mathrm{T}})$, then since rank stays the same for transposes, (apologies that this fact is not yet proved), we have $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

50 We are given $AB = I$ which has rank n. Then rank$(AB) \leq$ rank(A) forces rank$(A) = n$. This means that A is invertible. The right-inverse B is also a left-inverse: $BA = I$ and $B = A^{-1}$.

51 Certainly A and B have at most $\operatorname{rank} 2$. Then their product AB has at most $\operatorname{rank} 2$. Since BA is 3 by 3, it cannot be I even if $AB = I$.

52 (a) A and B will both have the same nullspace and row space as the R they share.

(b) A equals an ***invertible*** matrix times B, when they share the same R. A key fact!

53 $A = (\text{pivot columns})(\text{nonzero rows of } R) = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} +$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$. $B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{matrix} \text{columns} \\ \text{times rows} \end{matrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$

54 If $c = 1, R = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_2, x_3, x_4 free. If $c \neq 1, R = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

has x_3, x_4 free. Special solutions in $N = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (for $c = 1$) and $N =$

$\begin{bmatrix}-2 & -2\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}$ (for $c \neq 1$). If $c = 1, R = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ and x_1 free; if $c = 2, R = \begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}$ and x_2 free; $R = I$ if $c \neq 1, 2$. Special solutions in $N = \begin{bmatrix}1\\ 0\end{bmatrix}$ $(c = 1)$ or $N = \begin{bmatrix}2\\ 1\end{bmatrix}$ $(c = 2)$ or $N = 2$ by 0 empty matrix.

55 $A = \begin{bmatrix}I & I\end{bmatrix}$ has $N = \begin{bmatrix}I\\ -I\end{bmatrix}$; $B = \begin{bmatrix}I & I\\ 0 & 0\end{bmatrix}$ has the same N; $C = \begin{bmatrix}I & I & I\end{bmatrix}$ has $N = \begin{bmatrix}-I & -I\\ I & 0\\ 0 & I\end{bmatrix}$.

56 $\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix} =$ (pivot column) (first row) $= \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$

57 The m by n matrix Z has r ones to start its main diagonal. Otherwise Z is all zeros.

58 $R = \begin{bmatrix}I & F\\ 0 & 0\end{bmatrix} = \begin{bmatrix}r \text{ by } r & r \text{ by } n-r\\ m-r \text{ by } r & m-r \text{ by } n-r\end{bmatrix}$; **rref**$(R^{\mathrm{T}}) = \begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}$; **rref**$(R^{\mathrm{T}}R) =$ same R

59 $R = \begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$ has $R^{\mathrm{T}}R = \begin{bmatrix}1 & 2 & 0\\ 2 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}$ and this matrix row reduces to $\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}R\\ \text{zero row}\end{bmatrix}$. Always $R^{\mathrm{T}}R$ has the same nullspace as R, so its row reduced form must be R with $n - m$ extra zero rows. R is determined by its nullspace and shape !

60 The *row-column reduced echelon form* is always $\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}$; I is r by r.

Problem Set 3.3, page 158

1 $\begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - \mathbf{2b}_1 \end{bmatrix}$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2,2,2)$ and $(4,5,3)$. **This is the plane** $b_3 + b_2 - 2b_1 = 0$ (!). The nullspace contains all combinations of $\boldsymbol{s}_1 = (-1,-1,1,0)$ and $\boldsymbol{s}_2 = (2,-2,0,1)$; $\boldsymbol{x}_{complete} = \boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$;

$\begin{bmatrix} R & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ gives the particular solution $\boldsymbol{x}_p = (4,-1,0,0)$.

2 $\begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - \mathbf{3b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - \mathbf{2b}_1 \end{bmatrix}$ Then $[\,R \quad \boldsymbol{d}\,] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; $\boldsymbol{C}(A) =$ line through $(2,6,4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2,1,0)$ and $\boldsymbol{s}_2 = (-3/2,0,1)$; particular solution $\boldsymbol{x}_p = \boldsymbol{d} = (5,0,0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

3 $\boldsymbol{x}_{\text{complete}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. The matrix is singular but the equations are still solvable; $\boldsymbol{b}$ is in the column space. Our particular solution has free variable $y = 0$.

4 $\boldsymbol{x}_{\text{complete}} = \boldsymbol{x}_p + \boldsymbol{x}_n = (\frac{1}{2}, 0, \frac{1}{2}, 0) + x_2(-3,1,0,0) + x_4(0,0,-2,1)$.

5 $\begin{bmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{bmatrix}$ solvable if $b_3 - 2b_1 - b_2 = 0$.

Back-substitution gives the particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ and the special solution to $A\boldsymbol{x} = \boldsymbol{0}$: $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix} = \boldsymbol{x}_p$

(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

7 $\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & b_2 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix}$ One more step gives $[\,0\ \ 0\ \ 0\ \ 0\,] =$ row 3 − 2 (row 2) + 4(row 1) **provided $\boldsymbol{b_3 - 2b_2 + 4b_1 = 0}$**.

8 (a) Every $\boldsymbol{b}$ is in $\boldsymbol{C}(A)$: ***independent rows***, only the zero combination gives $\boldsymbol{0}$.

(b) We need $b_3 = 2b_2$, because $(\text{row}\,3) - 2(\text{row}\,2) = \boldsymbol{0}$.

9 $L\begin{bmatrix} U & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{bmatrix}$

$= \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$; particular $\boldsymbol{x}_p = (-9, 0, 3, 0)$ means $-9(1, 2, 3) + 3(3, 8, 7) = (0, 6, -6)$. This is $A\boldsymbol{x}_p = \boldsymbol{b}$.

10 $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ has $\boldsymbol{x}_p = (2, 4, 0)$ and $\boldsymbol{x}_{\text{null}} = (c, c, c)$. Many possible A !

11 A 1 by 3 system has at least **two** free variables. But $\boldsymbol{x}_{\text{null}}$ in Problem 10 only has **one**.

12 (a) If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and $A\boldsymbol{x}_2 = \boldsymbol{b}$ then $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and also $\boldsymbol{x} = \boldsymbol{0}$ solve $A\boldsymbol{x} = \boldsymbol{0}$

(b) $A(2\boldsymbol{x}_1 - 2\boldsymbol{x}_2) = \boldsymbol{0}, A(2\boldsymbol{x}_1 - \boldsymbol{x}_2) = \boldsymbol{b}$

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be $\boldsymbol{x}_p$

(c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (length 2)

(d) The only "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \boldsymbol{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a ***free*** variable. The zero vector ***is not*** the only solution to $A\boldsymbol{x} = \mathbf{0}$. If this system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has ***infinitely many*** solutions.

15 If row 3 of U has no pivot, that is a ***zero row***. $U\boldsymbol{x} = \boldsymbol{c}$ is only solvable provided $\boldsymbol{c}_3 = 0$. $A\boldsymbol{x} = \boldsymbol{b}$ ***might not be solvable***, because U may have other zero rows needing more $c_i = 0$.

16 The largest rank is 3. Then there is a pivot in every ***row***. The solution ***always exists***. The column space is $\mathbf{R}^3$. An example is $A = [\, I \quad F\,]$ for any 3 by 2 matrix F.

17 The largest rank of a 6 by 4 matrix is 4. Then there is a pivot in every ***column***. The solution is ***unique*** (if there is a solution). The nullspace contains only the ***zero vector***. Then $\mathbf{R} = \mathbf{rref}(A) = \begin{bmatrix} I & (4 \text{ by } 4) \\ 0 & (2 \text{ by } 4) \end{bmatrix}$.

18 Rank $= 2$; rank $= 3$ unless $q = 2$ (then rank $= 2$). Transpose has the same rank!

19 Both matrices A have rank 2. Always $A^{\mathrm{T}}A$ and AA^{T} have **the same rank** as A.

20 $A = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$; $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \mathbf{2} & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}$.

21 (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The second equation in part (b) removed one special solution from the nullspace.

22 If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and also $A\boldsymbol{x}_2 = \boldsymbol{b}$ then $A(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \mathbf{0}$ and we can add $\boldsymbol{x}_1 - \boldsymbol{x}_2$ to any solution of $A\boldsymbol{x} = \boldsymbol{B}$: the solution $\boldsymbol{x}$ is not unique. But there will be **no solution** to $A\boldsymbol{x} = \boldsymbol{B}$ if $\boldsymbol{B}$ is not in the column space.

23 For $A, q = 3$ gives rank 1, every other q gives rank 2. For $B, q = 6$ gives rank 1, every other q gives rank 2. These matrices cannot have rank 3.

24 (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} [\, x\,] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has 0 or 1 solutions, depending on $\boldsymbol{b}$ (b) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\, b\,]$ has infinitely many solutions for every b (c) There are 0 or ∞ solutions when A

has rank $r < m$ and $r < n$: the simplest example is a zero matrix. (d) *one* solution for all $\boldsymbol{b}$ when A is square and invertible (like $A = I$).

25 (a) $r < m$, always $r \le n$ (b) $r = m, r < n$ (c) $r < m, r = n$ (d) $r = m = n$.

26 $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \to R = \begin{bmatrix} \mathbf{1} & 0 & -2 \\ 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix} \to R = I$.

27 $R = I$ when A is square and invertible—so for a triangular matrix, all diagonal entries must be nonzero.

28 $\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{-1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$.

Free $x_2 = 0$ gives $\boldsymbol{x}_p = (-1, 0, 2)$ because the pivot columns contain I.

29 $[R\ \boldsymbol{d}] = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$ leads to $\boldsymbol{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $[R\ \boldsymbol{d}] = \begin{bmatrix} 1 & 0 & 0 & \mathbf{-1} \\ 0 & 0 & 1 & \mathbf{2} \\ 0 & 0 & 0 & \mathbf{5} \end{bmatrix}$:

this has no solution because of the 3rd equation

30 $\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & \mathbf{-4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$; $\begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$; $\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

31 For $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$, the only solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. B cannot exist since 2 equations in 3 unknowns cannot have a unique solution.

32 $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix}$ factors into $LU = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the rank is $r = 2$. The special solution to $A\boldsymbol{x} = \mathbf{0}$ and $U\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{s} = (-7, 2, 1)$. Since

$\boldsymbol{b} = (1,3,6,5)$ is also the last column of A, a particular solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $(0,0,1)$ and the complete solution is $\boldsymbol{x} = (0,0,1) + c\boldsymbol{s}$. (Or use the particular solution $\boldsymbol{x}_p = (7,-2,0)$ with free variable $x_3 = 0$.)

For $\boldsymbol{b} = (1,0,0,0)$ elimination leads to $U\boldsymbol{x} = (1,-1,0,1)$ and the fourth equation is $0 = 1$. No solution for this $\boldsymbol{b}$.

33 If the complete solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix}$ then $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

34 (a) If $\boldsymbol{s} = (2,3,1,0)$ is the only special solution to $A\boldsymbol{x} = \boldsymbol{0}$, the complete solution is $\boldsymbol{x} = c\boldsymbol{s}$ (a line of solutions). The rank of A must be $4 - 1 = 3$.

(b) The fourth variable x_4 is ***not free*** in $\boldsymbol{s}$, and R must be $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) $A\boldsymbol{x} = \boldsymbol{b}$ can be solved for all $\boldsymbol{b}$, because A and R have ***full row rank*** $r = 3$.

35 For the $-1,2,-1$ matrix K(9 by 9) and constant right side $\boldsymbol{b} = (10,\cdots,10)$, the solution $\boldsymbol{x} = K^{-1}\boldsymbol{b} = (45,80,105,120,125,120,105,80,45)$ rises and falls along the parabola $x_i = 50i - 5i^2$. (A formula for K^{-1} is later in the text.)

36 If $A\boldsymbol{x} = \boldsymbol{b}$ and $C\boldsymbol{x} = \boldsymbol{b}$ have the same solutions, A and C have the same shape and the same nullspace (take $\boldsymbol{b} = \boldsymbol{0}$). If $\boldsymbol{b} =$ column 1 of A, $\boldsymbol{x} = (1,0,\ldots,0)$ solves $A\boldsymbol{x} = \boldsymbol{b}$ so it solves $C\boldsymbol{x} = \boldsymbol{b}$. Then A and C share column 1. Other columns too: $A = C$!

37 The column space of R (m by n with rank r) spanned by its r pivot columns (the first r columns of an m by m identity matrix).

Problem Set 3.4, page 175

1 $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$ gives $c_3 = c_2 = c_1 = 0$. So those 3 column vectors are independent. But $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \\ c \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is solved by $\boldsymbol{c} = (1, 1, -4, 1)$. Then $\boldsymbol{v}_1 + \boldsymbol{v}_2 - 4\boldsymbol{v}_3 + \boldsymbol{v}_4 = \mathbf{0}$ (dependent).

2 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors in $\mathbf{R}^4$ are on the plane $(1, 1, 1, 1) \cdot \boldsymbol{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column $1 = \mathbf{0}$; if $d = 0$ then $b(\text{column } 1) - a(\text{column } 2) = \mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

4 $U\boldsymbol{x} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives $z = 0$ then $y = 0$ then $x = 0$ (by back substitution). A square triangular matrix has independent columns (invertible matrix) *when its diagonal has no zeros*.

5 (a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & -1 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{bmatrix}$: invertible $\Rightarrow$ independent columns.

(b) $\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$; $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ columns add to $\mathbf{0}$.

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A. This is because $EA = U$ for the matrix E that subtracts 2 times row 1 from row 4. So A and U have the same nullspace (same dependencies of columns).

7 The sum $\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \mathbf{0}$ because $(\boldsymbol{w}_2 - \boldsymbol{w}_3) - (\boldsymbol{w}_1 - \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) = \mathbf{0}$. So the differences are ***dependent*** and the difference matrix is singular: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

8 If $c_1(\boldsymbol{w}_2 + \boldsymbol{w}_3) + c_2(\boldsymbol{w}_1 + \boldsymbol{w}_3) + c_3(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\boldsymbol{w}_1 + (c_1 + c_3)\boldsymbol{w}_2 + (c_1 + c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

(changing -1's to 1's for the matrix A in solution **7** above makes A invertible.)

9 (a) The four vectors in $\mathbf{R}^3$ are the columns of a 3 by 4 matrix A. There is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$ because there is at least one free variable (b) Two vectors are dependent if $[\,\boldsymbol{v}_1 \;\; \boldsymbol{v}_2\,]$ has rank 0 or 1. (OK to say "they are on the same line" or "one is a multiple of the other" but ***not*** "$\boldsymbol{v}_2$ is a multiple of $\boldsymbol{v}_1$" —since $\boldsymbol{v}_1$ might be $\mathbf{0}$.) (c) A nontrivial combination of $\boldsymbol{v}_1$ and $\mathbf{0}$ gives $\mathbf{0}$: $0\boldsymbol{v}_1 + 3(0, 0, 0) = \mathbf{0}$.

10 The plane is the nullspace of $A = [\,1 \;\; 2 \;\; -3 \;\; -1\,]$. Three free variables give three independent solutions $(x, y, z, t) = (2, -1, 0, 0)$ and $(3, 0, 1, 0)$ and $(1, 0, 0, 1)$. Combinations of those special solutions give more solutions (all solutions).

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. ***False***. The zero vector is always in the row space.

13 The column space and row space of A and U all have the same dimension = 2. ***The row spaces of A and U are the same***, because the rows of U are combinations of the rows of A (and vice versa!).

14 $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) + \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$ and $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$. The two pairs ***span*** the same space. They are a basis when $\boldsymbol{v}$ and $\boldsymbol{w}$ are ***independent***.

15 The n independent vectors span a space of dimension n. They are a ***basis*** for that space. If they are the columns of A then m is ***not less*** than n $(m \geq n)$. ***Invertible*** if $m = n$.

16 These bases are not unique! (a) $(1, 1, 1, 1)$ for the space of all constant vectors (c, c, c, c) (b) $(1, -1, 0, 0), (1, 0, -1, 0), (1, 0, 0, -1)$ for the space of vectors with sum of components = 0 (c) $(1, -1, -1, 0), (1, -1, 0, -1)$ for the space perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$ (d) The columns of I are a basis for its column space, the empty set is a basis (by convention) for $\boldsymbol{N}(I) = \boldsymbol{Z} = \{\text{zero vector}\}$.

17 The column space of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$ is $\mathbf{R}^2$ so take any bases for $\mathbf{R}^2$; (row 1 and row 2) or (row 1 and row $1 +$ row 2) or (row 1 and $-$ row 2) are bases for the row space of U.

18 (a) The 6 vectors ***might not*** span $\mathbf{R}^4$ (b) The 6 vectors ***are not*** independent (c) Any four ***might be*** a basis.

19 n independent columns $\Rightarrow$ rank n. Columns span $\mathbf{R}^m \Rightarrow$ rank m. Columns are basis for $\mathbf{R}^m \Rightarrow$ ***rank*** $= m = n$. The rank counts the number of ***independent*** columns.

20 One basis is $(2, 1, 0)$, $(-3, 0, 1)$. A basis for the intersection with the xy plane is $(2, 1, 0)$. The normal vector $(1, -2, 3)$ is a basis for the line perpendicular to the plane.

21 (a) The only solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \mathbf{0}$ because ***the columns are independent*** (b) $A\boldsymbol{x} = \boldsymbol{b}$ is solvable because ***the columns span*** $\mathbf{R}^5$. Key point: A basis gives exactly one solution for every $\boldsymbol{b}$.

22 (a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in **S**.

23 Columns 1 and 2 are bases for the (**different**) column spaces of A and U; rows 1 and 2 are bases for the (**equal**) row spaces of A and U; $(1, -1, 1)$ is a basis for the (**equal**) nullspaces.

24 (a) ***False*** $A = [\,1 \;\; 1\,]$ has dependent columns, independent row (b) ***False*** Column space $\neq$ row space for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (c) ***True***: Both dimensions $= 2$ if A is invertible, dimensions $= 0$ if $A = 0$, otherwise dimensions $= 1$ (d) ***False***, columns may be dependent, in that case not a basis for $\boldsymbol{C}(A)$.

25 A has rank 2 if $c = 0$ and $d = 2$; $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ has rank 2 except when $c = d$ or $c = -d$.

26 (a) Basis for all diagonal matrices: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Add $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ = basis for symmetric matrices.

(c) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

These are simple bases (among many others) for (a) diagonal matrices (b) symmetric matrices (c) skew-symmetric matrices. The dimensions are $3, 6, 3$.

27 I, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$;

echelon matrices do ***not*** form a subspace; they ***span*** the upper triangular matrices (not every U is an echelon matrix).

28 $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

29 (a) The invertible matrices span the space of all 3 by 3 matrices (b) The rank one matrices also span the space of all 3 by 3 matrices (c) I by itself spans the space of all multiples cI.

30 $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$. **Dimension** $= 4$.

31 (a) $y(x) =$ constant C (b) $y(x) = 3x$. (c) $y(x) = 3x + C = y_p + y_n$ solves $y' = 3$.

32 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

33 (a) $y(x) = e^{2x}$ is a basis for all solutions to $y' = 2y$ (b) $y = x$ is a basis for all solutions to $dy/dx = y/x$ (First-order linear equation $\Rightarrow$ 1 basis function in solution space).

34 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

35 Basis 1, x, x^2, x^3, for cubic polynomials; basis $x-1$, x^2-1, x^3-1 for the subspace with $p(1) = 0$.

36 Basis for $\mathbf{S}$: $(1,0,-1,0)$, $(0,1,0,0)$, $(1,0,0,-1)$; basis for $\mathbf{T}$: $(1,-1,0,0)$ and $(0,0,2,1)$; $\mathbf{S} \cap \mathbf{T} =$ multiples of $(3,-3,2,1) =$ nullspace for 3 equations in $\mathbf{R}^4$ has dimension 1.

37 The subspace of matrices that have $AS = SA$ has dimension ***three***. The 3 numbers a, b, c can be chosen independently in A.

38 (a) No, 2 vectors don't span $\mathbf{R}^3$ (b) No, 4 vectors in $\mathbf{R}^3$ are dependent (c) Yes, a basis (d) No, these three vectors are dependent

39 If the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A: no solution to $A\boldsymbol{x} = \boldsymbol{b}$. If $[\,A \;\; \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent (rank 4), ***b is*** a combination of those columns. In this case $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

40 (a) The functions $y = \sin x$, $y = \cos x$, $y = e^x$, $y = e^{-x}$ are a basis for solutions to $d^4y/dx^4 = y(x)$.

(b) A particular solution to $d^4y/dx^4 = y(x)+1$ is $y(x) = -\mathbf{1}$. The complete solution is $y(x) = -1 + c_1 \sin x + c_2 \cos x + c_3 e^x + c_4 e^{-x}$ (or use another basis for the nullspace of the 4th derivative).

41 $\boldsymbol{I} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$. The six P's are dependent.

Those five are independent: The 4th has $P_{11} = 1$ and cannot be a combination of the others. Then the 2nd cannot be (from $P_{32} = 1$) and also 5th ($P_{32} = 1$). Continuing, a nonzero combination of all five could not be zero. Further challenge: How many independent 4 by 4 permutation matrices?

42 The dimension of $\boldsymbol{S}$ spanned by all rearrangements of $\boldsymbol{x}$ is (a) zero when $\boldsymbol{x} = \mathbf{0}$ (b) one when $\boldsymbol{x} = (1,1,1,1)$ (c) three when $\boldsymbol{x} = (1,1,-1,-1)$ because all rearrangements of this $\boldsymbol{x}$ are perpendicular to $(1,1,1,1)$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** dim $\boldsymbol{S} = 2$. I owe this nice problem to Mike Artin—the answers are the same in higher dimensions: $0, 1, n-1, n$.

43 The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\mathbf{0}$. **To be proved**: All coefficients $=$ zero.

Key idea: In that combination giving $\mathbf{0}$, the part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$. So the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now the combination giving $\mathbf{0}$ uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \mathbf{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

44 The inputs to multiplication by an m by n matrix fill $\mathbf{R}^n$: dimension n. The outputs (column space!) have dimension r. The nullspace has $n - r$ special solutions. The formula becomes $r + (n - r) = n$.

45 If the left side of $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$ is greater than n, then $\dim(\mathbf{V} \cap \mathbf{W})$ must be greater than zero. So $\mathbf{V} \cap \mathbf{W}$ contains nonzero vectors.

Oh here is a more basic approach: Put a basis for **V** and then a basis for **W** in the columns of a matrix A. Then A has more columns than rows and there is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$. That $\boldsymbol{x}$ gives a combination of the **V** columns $=$ a combination of the **W** columns.

46 If $A^2 =$ zero matrix, this says that each column of A is in the nullspace of A. If the column space has dimension r, the nullspace has dimension $10 - r$. So we must have $r \leq 10 - r$ and this leads to $r \leq 5$.

Problem Set 3.5, page 190

1 (a) Row and column space dimensions $= 5$, nullspace dimension $= 4$, $\dim(\boldsymbol{N}(A^{\mathrm{T}}))$ $= 2$ sum $5 + 5 + 4 + 2 = 16 = m + n$

(b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$.

2 A: Row space basis $=$ row $1 = (1, 2, 4)$; nullspace $(-2, 1, 0)$ and $(-4, 0, 1)$; column space basis $=$ column $1 = (1, 2)$; left nullspace $(-2, 1)$. B: Row space basis $=$ both rows $= (1, 2, 4)$ and $(2, 5, 8)$; column space basis $=$ two columns $= (1, 2)$ and $(2, 5)$; nullspace $(-4, 0, 1)$; left nullspace basis is empty because the space contains only $\boldsymbol{y} = \mathbf{0}$: the rows of B are independent.

3 Row space basis $=$ first two rows of U; column space basis $=$ pivot columns (of A not U) $= (1, 1, 0)$ and $(3, 4, 1)$; nullspace basis $(1, 0, 0, 0, 0)$, $(0, 2, -1, 0, 0)$, $(0, 2, 0, -2, 1)$; left nullspace $(1, -1, 1) =$ last row of $E^{-1} = L$.

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r + (n - r)$ must be 3 (c) $\begin{bmatrix} 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 9 & -3 \\ 3 & -1 \end{bmatrix}$

(e) ***Impossible*** Row space $=$ column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension. Section 4.1 will prove $\boldsymbol{N}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

5 $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ has those rows spanning its row space. $B = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$ has the same rows spanning its nullspace and $AB^{\mathrm{T}} = 0$.

6 A: dim $\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}$: Rows $(0, 3, 3, 3)$ and $(0, 1, 0, 1)$; columns $(3, 0, 1)$ and $(3, 0, 0)$; nullspace $(1, 0, 0, 0)$ and $(0, -1, 0, 1)$; $\boldsymbol{N}(A^{\mathrm{T}})$ $(0, 1, 0)$. B: dim $\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{2}$ Row space (1), column space $(1, 4, 5)$, nullspace: empty basis, $\boldsymbol{N}(A^{\mathrm{T}})$ $(-4, 1, 0)$ and $(-5, 0, 1)$.

7 Invertible 3 by 3 matrix A: row space basis $=$ column space basis $= (1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; nullspace basis and left nullspace basis are *empty*. Matrix $B = \begin{bmatrix} A & A \end{bmatrix}$: row space basis $(1, 0, 0, 1, 0, 0)$, $(0, 1, 0, 0, 1, 0)$ and $(0, 0, 1, 0, 0, 1)$; column space basis

$(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis $(-1,0,0,1,0,0)$ and $(0,-1,0,0,1,0)$ and $(0,0,-1,0,0,1)$; left nullspace basis is empty.

8 $\begin{bmatrix} I & 0 \end{bmatrix}$ and $\begin{bmatrix} I & I; & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \end{bmatrix} = 3$ by 2 have ***row space dimensions*** $= 3, 3, 0 =$ column space dimensions; ***nullspace dimensions*** $2, 3, 2$; ***left nullspace dimensions*** $0, 2, 3$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same

(b) Same column space and left nullspace. Same rank (dimension of column space).

10 For **rand** (3), almost surely rank$= 3$, nullspace and left nullspace contain only $(0,0,0)$. For **rand** $(3,5)$ the rank is almost surely 3 and the dimension of the nullspace is 2.

11 (a) No solution means that $r < m$. Always $r \le n$. Can't compare m and n here.

(b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \boldsymbol{0}$ is in both $\boldsymbol{N}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$.

13 (a) ***False***: Usually row space $\neq$ column space (they do not have the same dimension!)

(b) ***True***: A and $-A$ have the same four subspaces

(c) ***False*** (choose A and B same size and invertible: then they have the same four subspaces)

14 Row space basis can be the nonzero rows of U: $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$ as for U; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (happen to have $\boldsymbol{C}(A) = \boldsymbol{C}(U) = \mathbf{R}^3$); left nullspace has empty basis.

15 After a row exchange, the row space and nullspace stay the same; $(2,1,3,4)$ is in the new left nullspace after the row exchange.

16 If $A\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$. So $\boldsymbol{v} = \boldsymbol{0}$.

17 Row space $= yz$ plane; column space $= xy$ plane; nullspace $= x$ axis; left nullspace $= z$ axis. For $I + A$: Row space $=$ column space $= \mathbf{R}^3$, both nullspaces contain only the zero vector.

18 Row $3-2$ row $2+$ row $1=$ zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

19 (a) Elimination on $A\boldsymbol{x}=\mathbf{0}$ leads to $0=b_3-b_2-b_1$ so $(-1,-1,1)$ is in the left nullspace. (b) 4 by 3: Elimination leads to $b_3-2b_1=0$ and $b_4+b_2-4b_1=0$, so $(-2,0,1,0)$ and $(-4,1,0,1)$ are in the left nullspace. ***Why***? Those vectors multiply the matrix to give ***zero rows*** in $\boldsymbol{v}A$. Section 4.1 will show another approach: $A\boldsymbol{x}=\boldsymbol{b}$ is solvable ($\boldsymbol{b}$ is in $\boldsymbol{C}(A)$) exactly when $\boldsymbol{b}$ is orthogonal to the left nullspace.

20 (a) Special solutions $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ are perpendicular to the rows of R (and rows of ER). (b) $A^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$ has 1 independent solution $=$ last row of E^{-1}. ($E^{-1}A=R$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank <2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}+\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

22 $A=\begin{bmatrix} & \\ \boldsymbol{u} & \boldsymbol{w} \\ & \end{bmatrix}\begin{bmatrix}\boldsymbol{v}^{\mathrm{T}}\\ \boldsymbol{z}^{\mathrm{T}}\end{bmatrix}=\begin{bmatrix}1&2\\2&2\\4&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix}=\begin{bmatrix}3&2\\4&2\\5&1\end{bmatrix}$ $\boldsymbol{u},\boldsymbol{w}$ span column space; $\boldsymbol{v},\boldsymbol{z}$ span row space

23 As in Problem 22: Row space basis $(3,0,3),(1,1,2)$; column space basis $(1,4,2)$, $(2,5,7)$; the rank of (3 by 2) times (2 by 3) cannot be larger than the rank of either factor, so rank ≤ 2 and the 3 by 3 product is not invertible.

24 $A^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{d}$ puts $\boldsymbol{d}$ in the ***row space*** of A; unique solution if the ***left nullspace*** (nullspace of A^{T}) contains only $\boldsymbol{y}=\mathbf{0}$.

25 (a) ***True*** (A and A^{T} have the same rank) (b) ***False*** $A=[\,1\ \ 0\,]$ and A^{T} have very different left nullspaces (c) ***False*** (A can be invertible and unsymmetric even if $\boldsymbol{C}(A)=\boldsymbol{C}(A^{\mathrm{T}})$) (d) ***True*** (The subspaces for A and $-A$ are always the same. If $A^{\mathrm{T}}=A$ or $A^{\mathrm{T}}=-A$ they are also the same for A^{T})

26 Choose $d=bc/a$ to make $\left[\begin{smallmatrix}\mathbf{a}&\mathbf{b}\\\mathbf{c}&\mathbf{d}\end{smallmatrix}\right]$ a rank-1 matrix. Then the row space has basis (a,b) and the nullspace has basis $(-b,a)$. Those two vectors are perpendicular !

27 B and C (checkers and chess) both have rank 2 if $p\neq 0$. Row 1 and 2 are a basis for the row space of C, $B^{\mathrm{T}}\boldsymbol{y}=\mathbf{0}$ has 6 special solutions with -1 and 1 separated by a zero;

$\boldsymbol{N}(C^{\mathrm{T}})$ has $(-1,0,0,0,0,0,0,1)$ and $(0,-1,0,0,0,0,1,0)$ and columns $3,4,5,6$ of I; $\boldsymbol{N}(C)$ is a challenge: one vector in $\boldsymbol{N}(C)$ is $(1,0,\ldots,0,-1)$.

28 $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1$. (Need to specify the five moves).

29 The subspaces for $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B = cA$ with $c \neq 0$.

30 (a) $AX = 0$ if each column of X is a multiple of $(1,1,1)$; $\dim(\text{nullspace}) = 3$. (b) If $AX = B$ then all columns of B add to zero; dimension of the B's $= 6$. (c) $3 + 6 = \dim(M^{3\times 3}) = 9$ entries in a 3 by 3 matrix.

31 The key is equal row spaces. First row of A = combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F = G$.