

Problem Set 10.1, page 459

1 $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$; nullspace contains $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^T \mathbf{y} = \mathbf{0}$ for $\mathbf{y} = (1, -1, 1)$; current along edge 1, edge 3, back on edge 2 (full loop).

3 Elimination on $b_1[A \ \mathbf{b}] = \begin{bmatrix} -1 & 1 & 0 & b_1 \\ -1 & 0 & 1 & b_2 \\ 0 & -1 & 1 & b_3 \end{bmatrix}$ leads to $[U \ \mathbf{c}] =$

$\begin{bmatrix} -1 & 1 & 0 & b_1 \\ 0 & -1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 + b_1 \end{bmatrix}$. The nonzero rows of U come from edges 1 and 3 in a tree. The zero row comes from the loop (all 3 edges).

4 For the matrix in Problem 3, $A\mathbf{x} = \mathbf{b}$ is solvable for $\mathbf{b} = (1, 1, 0)$ and not solvable for $\mathbf{b} = (1, 0, 0)$. For solvable \mathbf{b} (in the column space), \mathbf{b} must be orthogonal to $\mathbf{y} = (1, -1, 1)$; that combination of rows is the zero row, and $b_1 - b_2 + b_3 = 0$ is the third equation after elimination.

5 Kirchhoff's Current Law $A^T \mathbf{y} = \mathbf{f}$ is solvable for $\mathbf{f} = (1, -1, 0)$ and not solvable for $\mathbf{f} = (1, 0, 0)$; \mathbf{f} must be orthogonal to $(1, 1, 1)$ in the nullspace: $f_1 + f_2 + f_3 = 0$.

6 $A^T A \mathbf{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \mathbf{f}$ produces $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\mathbf{x} = 1, -1, 0$ and currents $-A\mathbf{x} = 2, 1, -1$; \mathbf{f} sends 3 units from node 2 into node 1.

7 $A^T \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} A = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}$; $\mathbf{f} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ yields $\mathbf{x} = \begin{bmatrix} 5/4 \\ 1 \\ 7/8 \end{bmatrix} + \text{any } \begin{bmatrix} c \\ c \\ c \end{bmatrix}$;
potentials $\mathbf{x} = \frac{5}{4}, 1, \frac{7}{8}$ and currents $-CA\mathbf{x} = \frac{1}{4}, \frac{3}{4}, \frac{1}{4}$.

$$\mathbf{8} \quad A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \text{ leads to } \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \text{ solving } A^T \mathbf{y} = \mathbf{0}.$$

- 9** Elimination on $A\mathbf{x} = \mathbf{b}$ always leads to $\mathbf{y}^T \mathbf{b} = 0$ in the zero rows of U and R : $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ (those \mathbf{y} 's are from Problem 8 in the left nullspace). This is Kirchhoff's *Voltage Law* around the two *loops*.

$$\mathbf{10} \quad \text{The echelon form of } A \text{ is } U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{The nonzero rows of } U \text{ keep} \\ \text{edges } 1, 2, 4. \text{ Other spanning trees} \\ \text{from edges, } 1, 2, 5; 1, 3, 4; 1, 3, 5; \\ 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 4, 5. \end{array}$$

$$\mathbf{11} \quad A^T A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \quad \begin{array}{l} \text{diagonal entry} = \text{number of edges into the node} \\ \text{the trace is 2 times the number of nodes} \\ \text{off-diagonal entry} = -1 \text{ if nodes are connected} \\ A^T A \text{ is the } \mathbf{\text{graph Laplacian}}, A^T C A \text{ is } \mathbf{\text{weighted by } C} \end{array}$$

- 12** (a) The nullspace and rank of $A^T A$ and A are always the same (b) $A^T A$ is always positive semidefinite because $\mathbf{x}^T A^T A \mathbf{x} = \|A\mathbf{x}\|^2 \geq 0$. Not positive definite because rank is only 3 and $(1, 1, 1, 1)$ is in the nullspace (c) Real eigenvalues all ≥ 0 because positive semidefinite.

$$\mathbf{13} \quad A^T C A \mathbf{x} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \begin{array}{l} \text{gives four potentials } \mathbf{x} = (\frac{5}{12}, \frac{1}{6}, \frac{1}{6}, 0) \\ \text{I grounded } x_4 = 0 \text{ and solved for } \mathbf{x} \\ \text{currents } \mathbf{y} = -C A \mathbf{x} = (\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2}) \end{array}$$

- 14** $A^T C A \mathbf{x} = \mathbf{0}$ for $\mathbf{x} = c(1, 1, 1, 1) = (c, c, c, c)$. If $A^T C A \mathbf{x} = \mathbf{f}$ is solvable, then \mathbf{f} in the column space (= row space by symmetry) must be orthogonal to \mathbf{x} in the nullspace: $f_1 + f_2 + f_3 + f_4 = 0$.

- 15** The number of loops in this connected graph is $n - m + 1 = 7 - 7 + 1 = 1$.
What answer if the graph has two separate components (no edges between)?
- 16** Start from (4 nodes) – (6 edges) + (3 loops) = 1. If a new node connects to 1 old node, $5 - 7 + 3 = 1$. If the new node connects to 2 old nodes, a new loop is formed: $5 - 8 + 4 = 1$.
- 17** (a) 8 independent columns (b) f must be orthogonal to the nullspace so f 's add to zero (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.
- 18** A *complete graph* has $5 + 4 + 3 + 2 + 1 = 15$ edges. With n nodes that count is $1 + \dots + (n - 1) = n(n - 1)/2$. Tree has 5 edges.

Problem Set 10.2, page 472

1 Det $A_0^T C_0 A_0 = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$ is by direct calculation. Set $c_4 = 0$ to find $\det A_1^T C_1 A_1 = c_1 c_2 c_3$.

2 $(A_1^T C_1 A_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1^{-1} & & \\ & c_2^{-1} & \\ & & c_3^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} =$

$$\begin{bmatrix} c_1^{-1} & c_1^{-1} & c_1^{-1} \\ c_1^{-1} & c_1^{-1} + c_2^{-1} & c_1^{-1} + c_2^{-1} \\ c_1^{-1} & c_1^{-1} + c_2^{-1} & c_1^{-1} + c_2^{-1} + c_3^{-1} \end{bmatrix}.$$

- 3** The rows of the free-free matrix in equation (9) add to $[0 \ 0 \ 0]$ so the right side needs $f_1 + f_2 + f_3 = 0$. $f = (-1, 0, 1)$ gives $c_2 u_1 - c_2 u_2 = -1$, $c_3 u_2 - c_3 u_3 = -1$, $0 = 0$. Then $u_{\text{particular}} = (-c_2^{-1} - c_3^{-1}, -c_3^{-1}, 0)$. Add any multiple of $u_{\text{nullspace}} = (1, 1, 1)$.

4 $\int -\frac{d}{dx} \left(c(x) \frac{du}{dx} \right) dx = - \left[c(x) \frac{du}{dx} \right]_0^1 = 0$ (bdry cond) so we need $\int f(x) dx = 0$.

- 5** $-\frac{dy}{dx} = f(x)$ gives $y(x) = C - \int_0^x f(t)dt$. Then $y(1) = 0$ gives $C = \int_0^1 f(t)dt$ and $y(x) = \int_x^1 f(t)dt$. If the load is $f(x) = 1$ then the displacement is $y(x) = 1 - x$.
- 6** Multiply $A_1^T C_1 A_1$ as columns of A_1^T times c 's times rows of A_1 . The first 3 by 3 "element matrix" $c_1 E_1 = [1 \ 0 \ 0]^T c_1 [1 \ 0 \ 0]$ has c_1 in the top left corner.
- 7** For 5 springs and 4 masses, the 5 by 4 A has two nonzero diagonals: all $a_{ii} = 1$ and $a_{i+1,i} = -1$. With $C = \text{diag}(c_1, c_2, c_3, c_4, c_5)$ we get $K = A^T C A$, symmetric tridiagonal with diagonal entries $K_{ii} = c_i + c_{i+1}$ and off-diagonals $K_{i+1,i} = -c_{i+1}$. With $C = I$ this K is the $-1, 2, -1$ matrix and $K(2, 3, 3, 2) = (1, 1, 1, 1)$ solves $Ku = \text{ones}(4, 1)$. (K^{-1} will solve $Ku = \text{ones}(4)$.)
- 8** The solution to $-u'' = 1$ with $u(0) = u(1) = 0$ is $u(x) = \frac{1}{2}(x - x^2)$. At $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ this gives $u = 2, 3, 3, 2$ (discrete solution in Problem 7) times $(\Delta x)^2 = 1/25$.
- 9** $-u'' = mg$ has complete solution $u(x) = A + Bx - \frac{1}{2}mgx^2$. From $u(0) = 0$ we get $A = 0$. From $u'(1) = 0$ we get $B = mg$. Then $u(x) = \frac{1}{2}mg(2x - x^2)$ at $x = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ equals $mg/6, 4mg/9, mg/2$. This $u(x)$ is *not* proportional to the discrete $u = (3mg, 5mg, 6mg)$ at the meshpoints. This imperfection is because the discrete problem uses a 1-sided difference, less accurate at the free end. Perfect accuracy is recovered by a centered difference (discussed on page 21 of my CSE textbook).
- 10** (added in later printing, changing **10-11** below into **11-12**). The solution in this fixed-fixed case is (2.25, 2.50, 1.75) so the second mass moves furthest.
- 11** The two graphs of 100 points are "discrete parabolas" starting at (0,0): symmetric around 50 in the fixed-fixed case, ending with slope zero in the fixed-free case.
- 12** Forward/backward/centered for du/dx has a big effect because that term has the large coefficient. MATLAB: $E = \text{diag}(\text{ones}(6, 1), 1)$; $K = 64 * (2 * \text{eye}(7) - E - E')$; $D = 80 * (E - \text{eye}(7))$; $(K + D) \setminus \text{ones}(7, 1)$; % forward; $(K - D) \setminus \text{ones}(7, 1)$; % backward; $(K + D/2 - D'/2) \setminus \text{ones}(7, 1)$; % centered is usually the best: more accurate

Problem Set 10.3, page 480

- 1** Eigenvalues $\lambda = 1$ and $.75$; $(A - I)\mathbf{x} = 0$ gives the steady state $\mathbf{x} = (.6, .4)$ with $A\mathbf{x} = \mathbf{x}$.
- 2** $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & .6 \end{bmatrix}$; $A^\infty = \begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.
- 3** $\lambda = 1$ and $.8$, $\mathbf{x} = (1, 0)$; 1 and $-.8$, $\mathbf{x} = (\frac{5}{9}, \frac{4}{9})$; $1, \frac{1}{4}$, and $\frac{1}{4}$, $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
- 4** A^T always has the eigenvector $(1, 1, \dots, 1)$ for $\lambda = 1$, because each row of A^T adds to 1. (Note again that many authors use row vectors multiplying Markov matrices. So they transpose our form of A .)
- 5** The steady state eigenvector for $\lambda = 1$ is $(0, 0, 1) =$ everyone is dead.
- 6** Add the components of $A\mathbf{x} = \lambda\mathbf{x}$ to find sum $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.
- 7** $(.5)^k \rightarrow 0$ gives $A^k \rightarrow A^\infty$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $\begin{matrix} a \leq 1 \\ .4 + .6a \geq 0 \end{matrix}$
- 8** If $P =$ cyclic permutation and $\mathbf{u}_0 = (1, 0, 0, 0)$ then $\mathbf{u}_1 = (0, 0, 1, 0)$; $\mathbf{u}_2 = (0, 1, 0, 0)$; $\mathbf{u}_3 = (1, 0, 0, 0)$; $\mathbf{u}_4 = \mathbf{u}_0$. The eigenvalues $1, i, -1, -i$ are all *on the unit circle*. This Markov matrix contains zeros; a *positive* matrix has *one* largest eigenvalue $\lambda = 1$.
- 9** M^2 is still nonnegative; $[1 \ \dots \ 1]M = [1 \ \dots \ 1]$ so multiply on the right by M to find $[1 \ \dots \ 1]M^2 = [1 \ \dots \ 1] \Rightarrow$ columns of M^2 add to 1.
- 10** $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $\mathbf{x}_1 = (b, 1 - a)$.
- 11** Last row $.2, .3, .5$ makes $A = A^T$; rows also add to 1 so $(1, \dots, 1)$ is also an eigenvector of A .
- 12** B has $\lambda = 0$ and $-.5$ with $\mathbf{x}_1 = (.3, .2)$ and $\mathbf{x}_2 = (-1, 1)$; A has $\lambda = 1$ so $A - I$ has $\lambda = 0$. $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t} \mathbf{x}_1 = c_1 \mathbf{x}_1$.
- 13** $\mathbf{x} = (1, 1, 1)$ is an eigenvector when the row sums are equal; $A\mathbf{x} = (.9, .9, .9)$

14 $(I-A)(I+A+A^2+\dots) = (I+A+A^2+\dots) - (A+A^2+A^3+\dots) = I$. This says that

$I + A + A^2 + \dots$ is $(I - A)^{-1}$. When $A = \begin{bmatrix} 0 & .5 \\ 1 & 0 \end{bmatrix}$, $A^2 = \frac{1}{2}I$, $A^3 = \frac{1}{2}A$, $A^4 = \frac{1}{4}I$

and the series adds to $\begin{bmatrix} 1 + \frac{1}{2} + \dots & \frac{1}{2} + \frac{1}{4} + \dots \\ 1 + \frac{1}{2} + \dots & 1 + \frac{1}{2} + \dots \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = (I - A)^{-1}$.

15 The first two A 's have $\lambda_{\max} < 1$; $\mathbf{p} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.

16 $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state (.3, .3, .4) and (30, 30, 40).

17 No, A has an eigenvalue $\lambda = 1$ and $(I - A)^{-1}$ does not exist.

18 The Leslie matrix on page 435 has $\det(A - \lambda I) = \det \begin{bmatrix} F_1 - \lambda & F_2 & F_3 \\ P_1 & -\lambda & 0 \\ 0 & P_2 & -\lambda \end{bmatrix} = -\lambda^3 +$

$F_1\lambda^2 + F_2P_1\lambda + F_3P_1P_2$. This is negative for large λ . It is positive at $\lambda = 1$ provided that $F_1 + F_2P_1 + F_3P_1P_2 > 1$. Under this key condition, $\det(A - \lambda I)$ must be zero at some λ between 1 and ∞ . That eigenvalue means that the population grows (under this condition connecting F 's and P 's reproduction and survival rates).

19 Λ times $X^{-1}\Delta X$ has the same diagonal as $X^{-1}\Delta X$ times Λ because Λ is diagonal.

20 If $B > A > 0$ and $A\mathbf{x} = \lambda_{\max}(A)\mathbf{x} > 0$ then $B\mathbf{x} > \lambda_{\max}(A)\mathbf{x}$ and $\lambda_{\max}(B) > \lambda_{\max}(A)$.
of $C =$ four components of $F\mathbf{c}$. Circulants are special!

Problem Set 10.4, page 489

- 1 Feasible set = line segment (6, 0) to (0, 3); minimum cost at (6, 0), maximum at (0, 3).
- 2 Feasible set has corners (0, 0), (6, 0), (2, 2), (0, 6). Minimum cost $2x - y$ at (6, 0).
- 3 Only two corners (4, 0, 0) and (0, 2, 0); let $x_i \rightarrow -\infty$, $x_2 = 0$, and $x_3 = x_1 - 4$.
- 4 From (0, 0, 2) move to $\mathbf{x} = (0, 1, 1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $\mathbf{x} = (1, 0, 1.5)$ with cost $5(1) + 8(1.5) = \$17$; $r = 1$ means more expensive.

- 5** Cost = 20 at start (4, 0, 0); keeping $x_1 + x_2 + 2x_3 = 4$ move to (3, 1, 0) with cost 18 and $r = -2$; or move to (2, 0, 1) with cost 17 and $r = -3$. Choose x_3 as entering variable and move to (0, 0, 2) with cost 14. Another step will reach (0, 4, 0) with minimum cost 12.
- 6** If we reduce the Ph.D. cost to \$1 or \$2 (below the student cost of \$3), the job will go to the Ph.D. with cost vector $\mathbf{c} = (2, 3, 8)$ the Ph.D. takes 4 hours ($x_1 + x_2 + 2x_3 = 4$) and charges \$8.

The teacher in the dual problem now has $y \leq 2, y \leq 3, 2y \leq 8$ as constraints $A^T \mathbf{y} \leq \mathbf{c}$ on the charge of y per problem. So the dual has maximum at $y = 2$. The dual cost is also \$8 for 4 problems and maximum = minimum.

- 7** $\mathbf{x} = (2, 2, 0)$ is a corner of the feasible set with $x_1 + x_2 + 2x_3 = 4$ and the new constraint $2x_1 + x_2 + x_3 = 6$. The cost of this corner is $\mathbf{c}^T \mathbf{x} = (5, 3, 8) \cdot (2, 2, 0) = 16$. Is this the minimum cost?

Compute the reduced cost r if $x_3 = 1$ enters (x_3 was previously zero). The two constraint equations now require $x_1 = 3$ and $x_2 = -1$. With $\mathbf{x} = (3, -1, 1)$ the new cost is $3.5 - 1.3 + 1.8 = 20$. This is higher than 16, so the original $\mathbf{x} = (2, 2, 0)$ was optimal.

Note that $x_3 = 1$ led to $x_2 = -1$ and a negative x_2 is not allowed. If x_3 reduced the cost (it didn't) we would not have used as much as $x_3 = 1$.

- 8** $\mathbf{y}^T \mathbf{b} \leq \mathbf{y}^T A \mathbf{x} = (A^T \mathbf{y})^T \mathbf{x} \leq \mathbf{c}^T \mathbf{x}$. The first inequality needed $\mathbf{y} \geq 0$ and $A \mathbf{x} - \mathbf{b} \geq 0$.

Problem Set 10.5, page 494

- 1** $\int_0^{2\pi} \cos((j+k)x) dx = \left[\frac{\sin((j+k)x)}{j+k} \right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos((j-k)x) dx = 0$
Notice $j - k \neq 0$ in the denominator. If $j = k$ then $\int_0^{2\pi} \cos^2 jx dx = \pi$.
- 2** Three integral tests show that $1, x, x^2 - \frac{1}{3}$ are orthogonal on the interval $[-1, 1]$:
 $\int_{-1}^1 (1)(x) dx = 0, \int_{-1}^1 (1)(x^2 - \frac{1}{3}) dx = 0, \int_{-1}^1 (x)(x^2 - \frac{1}{3}) dx = 0$. Then

$2x^2 = 2(x^2 - \frac{1}{3}) + 0(x) + \frac{2}{3}(1)$. Those coefficients $2, 0, \frac{2}{3}$ can come from integrating $f(x) = 2x^2$ times the 3 basis functions and dividing by their lengths squared—in other words using $\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}$ for functions (where \mathbf{b} is $f(x)$ and \mathbf{a} is 1 or x or $x^2 - \frac{1}{3}$) exactly as for vectors.

- 3** One example orthogonal to $\mathbf{v} = (1, \frac{1}{2}, \dots)$ is $\mathbf{w} = (2, -1, 0, 0, \dots)$ with $\|\mathbf{w}\| = \sqrt{5}$.
- 4** $\int_{-1}^1 (1)(x^3 - cx) dx = 0$ and $\int_{-1}^1 (x^2 - \frac{1}{3})(x^3 - cx) dx = 0$ for all c (odd functions). Choose c so that $\int_{-1}^1 x(x^3 - cx) dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^1 = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.
- 5** The integrals lead to the Fourier coefficients $a_1 = 0$, $b_1 = 4/\pi$, $b_2 = 0$.
- 6** From eqn. (3) $a_k = 0$ and $b_k = 4/\pi k$ (odd k). The square wave has $\|f\|^2 = 2\pi$. Then eqn. (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots)$. That infinite series equals $\pi^2/8$.
- 7** The $-1, 1$ odd square wave is $f(x) = x/|x|$ for $0 < |x| < \pi$. Its Fourier series in equation (8) is $4/\pi$ times $[\sin x + (\sin 3x)/3 + (\sin 5x)/5 + \dots]$. The sum of the first N terms has an interesting shape, close to the square wave except where the wave jumps between -1 and 1 . At those jumps, the Fourier sum spikes the wrong way to ± 1.09 (the *Gibbs phenomenon*) before it takes the jump with the true $f(x)$.

This happens for the Fourier sums of all functions with jumps. It makes shock waves hard to compute. You can see it clearly in a graph of the sum of 10 terms.

- 8** $\|\mathbf{v}\|^2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2$ so $\|\mathbf{v}\| = \sqrt{2}$; $\|\mathbf{v}\|^2 = 1 + a^2 + a^4 + \dots = 1/(1 - a^2)$ so $\|\mathbf{v}\| = 1/\sqrt{1 - a^2}$; $\int_0^{2\pi} (1 + 2\sin x + \sin^2 x) dx = 2\pi + 0 + \pi$ so $\|f\| = \sqrt{3\pi}$.
- 9** (a) $f(x) = (1 + \text{square wave})/2$ so the a 's are $\frac{1}{2}, 0, 0, \dots$ and the b 's are $2/\pi, 0, -2/3\pi, 0, 2/5\pi, \dots$. (b) $a_0 = \int_0^{2\pi} x dx / 2\pi = \pi$, all other $a_k = 0$, $b_k = -2/k$.
- 10** The integral from $-\pi$ to π or from 0 to 2π (or from any a to $a + 2\pi$) is over one complete period of the function. If $f(x)$ is periodic this changes $\int_0^{2\pi} f(x) dx$ to $\int_0^\pi f(x) dx + \int_{-\pi}^0 f(x) dx$. If $f(x)$ is **odd**, those integrals cancel to give $\int f(x) dx = 0$ over one period.
- 11** $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$; $\cos(x + \frac{\pi}{3}) = \cos x \cos \frac{\pi}{3} - \sin x \sin \frac{\pi}{3} = \frac{1}{2} \cos x - \frac{\sqrt{3}}{2} \sin x$.

$$\mathbf{12} \quad \frac{d}{dx} \begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin x \\ \cos x \\ -2\sin 2x \\ 2\cos 2x \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix}.$$

This shows the differentiation matrix.

13 The square pulse with $F(x) = 1/h$ for $-x \leq h/2 \leq x$ is an even function, so all sine coefficients b_k are zero. The average a_0 and the cosine coefficients a_k are

$$a_0 = \frac{1}{2\pi} \int_{-h/2}^{h/2} (1/h) dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi} \int_{-h/2}^{h/2} (1/h) \cos kx dx = \frac{2}{\pi kh} \left(\sin \frac{kh}{2} \right) \text{ which is } \frac{1}{\pi} \operatorname{sinc} \left(\frac{kh}{2} \right)$$

(introducing the sinc function $(\sin x)/x$). As h approaches zero, the number $x = kh/2$ approaches zero, and $(\sin x)/x$ approaches 1. So all those a_k approach $1/\pi$.

The limiting “delta function” contains an equal amount of all cosines: a very irregular function.

Problem Set 10.6, page 500

- 1** (x, y, z) has homogeneous coordinates (cx, cy, cz, c) for $c = 1$ and all $c \neq 0$.
- 2** For an affine transformation we also need T (origin), because $T(\mathbf{0})$ need not be $\mathbf{0}$ for affine T . Including this translation by $T(\mathbf{0})$, $(x, y, z, 1)$ is transformed to $xT(\mathbf{i}) + yT(\mathbf{j}) + zT(\mathbf{k}) + T(\mathbf{0})$.

$$\mathbf{3} \quad TT_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 6 & 8 & 1 \end{bmatrix} \text{ is translation along } (1, 6, 8).$$

- 4** $S = \operatorname{diag}(c, c, c, 1)$; row 4 of ST and TS is $1, 4, 3, 1$ and $c, 4c, 3c, 1$; use $\mathbf{v}TS$!

$$\mathbf{5} \quad S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix} \text{ for a 1 by 1 square, starting from an 8.5 by 11 page.}$$

$$\mathbf{6} \quad [x \ y \ z \ 1] \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ -1 & -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} = [x \ y \ z \ 1] \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ -2 & -2 & -4 & 1 \end{bmatrix}.$$

The first matrix translates by $(-1, -1, -2)$. The second matrix rescales by 2.

7 The three parts of Q in equation (1) are $(\cos \theta)I$ and $(1 - \cos \theta)\mathbf{a}\mathbf{a}^T$ and $-\sin \theta(\mathbf{a} \times)$. Then $Q\mathbf{a} = \mathbf{a}$ because $\mathbf{a}\mathbf{a}^T\mathbf{a} = \mathbf{a}$ (unit vector) and $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.

8 If $\mathbf{a}^T\mathbf{b} = 0$ and those three parts of Q (Problem 7) multiply \mathbf{b} , the results in $Q\mathbf{b}$ are $(\cos \theta)\mathbf{b}$ and $\mathbf{a}\mathbf{a}^T\mathbf{b} = \mathbf{0}$ and $(-\sin \theta)\mathbf{a} \times \mathbf{b}$. The component along \mathbf{b} is $(\cos \theta)\mathbf{b}$.

$$\mathbf{9} \quad \mathbf{n} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right) \text{ has } P = I - \mathbf{n}\mathbf{n}^T = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}. \text{ Notice } \|\mathbf{n}\| = 1.$$

$$\mathbf{10} \quad \text{We can choose } (0, 0, 3) \text{ on the plane and multiply } T_-PT_+ = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 6 & 6 & 3 & 9 \end{bmatrix}.$$

11 $(3, 3, 3)$ projects to $\frac{1}{3}(-1, -1, 4)$ and $(3, 3, 3, 1)$ projects to $(\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, 1)$. Row vectors!

12 The projection of a square onto a plane is a parallelogram (or a line segment). The sides of the square are perpendicular, but their projections may not be ($\mathbf{x}^T\mathbf{y} = 0$ but $(P\mathbf{x})^T(P\mathbf{y}) = \mathbf{x}^T P^T P \mathbf{y} = \mathbf{x}^T P \mathbf{y}$ may be nonzero).

13 That projection of a cube onto a plane produces a hexagon.

$$\mathbf{14} \quad (3, 3, 3)(I - 2\mathbf{n}\mathbf{n}^T) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} = \left(-\frac{11}{3}, -\frac{11}{3}, -\frac{1}{3} \right).$$

$$15 \quad (3, 3, 3, 1) \rightarrow (3, 3, 0, 1) \rightarrow \left(-\frac{7}{3}, -\frac{7}{3}, -\frac{8}{3}, 1\right) \rightarrow \left(-\frac{7}{3}, -\frac{7}{3}, \frac{1}{3}, 1\right).$$

16 Just subtracting vectors would give $\mathbf{v} = (x, y, z, 0)$ ending in 0 (not 1). In homogeneous coordinates, add a **vector** to a point.

17 Space is rescaled by $1/c$ because (x, y, z, c) is the same point as $(x/c, y/c, z/c, 1)$.

Problem Set 10.7, page 507

1 **Multiplying** n whole numbers gives an odd number only when *all n numbers are odd*.

This translates to multiplication (*mod 2*). Multiplying n 1's or 0's gives 1 only when all n numbers are 1.

2 **Adding** n whole numbers gives an odd number only when the n numbers include an *odd number of odd numbers*. For addition of 1's and 0's (*mod 2*), the answer is odd when the number of 1's is odd.

3 (a) We are given that $y_1 - x_1$ and $y_2 - x_2$ are both divisible by p . Then their sum $y_1 + y_2 - x_1 - x_2$ is divisible by p .

(b) $5 \equiv 2 \pmod{3}$ and $8 \equiv 2 \pmod{3}$ add to $13 \equiv 4 \pmod{3}$. The number 1 is smaller than 4 and $13 \equiv 1 \pmod{3}$.

5 If $y - x$ is divisible by p then $x - y$ is also divisible by p . In other words, if $y - x = mp$ then $x - y = (-m)p$.

6 $A = \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}$ is an invertible matrix but (*mod 5*) A becomes the zero matrix.

7 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ are invertible:

6 out of 16 possible 0-1 matrices.

8 Yes, $A\mathbf{x} = \mathbf{0} \pmod{11}$ says that every row of A is orthogonal to every \mathbf{x} in the nullspace (*mod 11*). But a basis for the usual $\mathbf{N}(A)$ could include vectors that are zero (*mod 11*).

- 9 For simplicity, number the letters as they appear in the message :

THISWHOLEBOOKISINCODE = 123/452/678/966/(10)34/3(11)(12)/6(13)8.

Multiply each block by this L to obtain Hill's cipher.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{Cipher} = 1\ 3\ 6/4\ 9\ 11/6\ 13\ 21/9\ 15\ 21/10\ 13\ 17/3\ 14\ 26/6\ 19\ 27.$$

If the cipher is *mod* p then replace each number by the correct number from 0 to $p - 1$.

To decode, first multiply by L^{-1} . Then what to do??

- 10 First you have to discover the block size (= matrix size) and also the matrix L itself. Start with a guess for the block size. Then the plaintext and the coded cipher tell you a series of matrix-vector products $L\mathbf{x} \equiv \mathbf{b}$. If the text is long enough (and the blocks are not too long) this is enough information to find L —or to show that the block size must be wrong, when there is no L that gets all correct blocks $L\mathbf{x} \equiv \mathbf{b}$.

The extra difficulty is to find the value of p .