Problem Set 5.1, page 254

1 $\det(2A) = 2^4 \det A = 8$; $\det(-A) = (-1)^4 \det A = \frac{1}{2}$; $\det(A^2) = \frac{1}{4}$; $\det(A^{-1}) = 2$.

2 $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = -\frac{1}{8}$ and $\det(-A) = (-1)^3 \det A = 1$; $\det(A^2) = 1$; $\det(A^{-1}) = -1$.

3 (a) *False*: $\det(I + I)$ is not $1 + 1$ (except when $n = 1$) (b) *True*: The product rule extends to ABC (use it twice) (c) *False*: $\det(4A)$ is $4^n \det A$
(d) *False*: $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $AB - BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is invertible.

4 Exchange rows 1 and 3 to show $|J_3| = -1$. Exchange rows 1 and 4, then rows 2 and 3 to show $|J_4| = 1$.

5 $|J_5| = 1$ by exchanging row 1 with 5 and row 2 with 4. $|J_6| = -1$, $|J_7| = -1$. Determinants $1, 1, -1, -1$ repeat in cycles of length 4 so the determinant of J_{101} is $+1$.

6 To prove Rule 6, multiply the zero row by $t = 2$. The determinant is multiplied by 2 (Rule 3) but the matrix is the same. So $2\det(A) = \det(A)$ and $\det(A) = 0$.

7 $\det(Q) = 1$ for rotation and $\det(Q) = 1 - 2\sin^2\theta - 2\cos^2\theta = -1$ for reflection.

8 $Q^{\mathrm{T}}Q = I \Rightarrow |Q^{\mathrm{T}}|\,|Q| = |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so its determinant can't blow up as $n \to \infty$.

9 $\det A = 1$ from two row exchanges . $\det B = 2$ (subtract rows 1 and 2 from row 3, then columns 1 and 2 from column 3). $\det C = 0$ (equal rows) even though $C = A + B$!

10 If the entries in every row add to zero, then $(1, 1, \ldots, 1)$ is in the nullspace: singular A has $\det = 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

11 $CD = -DC \Rightarrow \det CD = (-1)^n \det DC$ and *not* just $-\det DC$. If n is even then $\det CD = \det DC$ and we can have an invertible CD.

12 $\det(A^{-1})$ divides twice by $ad - bc$ (once for each row). This gives $\det A^{-1} = \dfrac{ad - bc}{(ad - bc)^2} = \dfrac{1}{ad - bc}$.

13 Pivots $1, 1, 1$ give determinant $= \mathbf{1}$; pivots $1, -2, -3/2$ give determinant $= \mathbf{3}$.

14 $\det(A) = \mathbf{36}$ and the 4 by 4 second difference matrix has det $= \mathbf{5}$.

15 The first determinant is $\mathbf{0}$, the second is $1 - 2t^2 + t^4 = (1 - t^2)^2$.

16 A singular rank one matrix has determinant $= 0$. The skew-symmetric K also has $\det K = 0$ (see $\#\mathbf{17}$): a skew-symmetric matrix K of odd order 3.

17 Any 3 by 3 skew-symmetric K has $\det(K^{\mathrm{T}}) = \det(-K) = (-1)^3\det(K)$. This is $-\det(K)$. But always $\det(K^{\mathrm{T}}) = \det(K)$. So we must have $\det(K) = 0$ for 3 by 3.

18 $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$ (to reach 2 by 2, eliminate a and a^2 in row 1 by column operations)—subtract a and a^2 times column 1 from columns 2 and 3. Factor out $b - a$ and $c - a$ from the 2 by 2: $(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b)$.

19 For triangular matrices, just multiply the diagonal entries: $\det(U) = 6, \det(U^{-1}) = \frac{1}{6}$, and $\det(U^2) = 36$. 2 by 2 matrix: $\det(U) = ad, \det(U^2) = a^2d^2$. If $ad \neq 0$ then $\det(U^{-1}) = 1/ad$.

20 $\det \begin{bmatrix} a-Lc & b-Ld \\ c-\ell a & d-\ell b \end{bmatrix}$ reduces to $(ad - bc)(1 - L\ell)$. The determinant changes if you do two row operations at once.

21 We can exchange rows using the three elimination steps in the problem, followed by multiplying row 1 by -1. So Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

22 $\det(A) = 3, \det(A^{-1}) = \frac{1}{3}, \det(A - \lambda I) = \lambda^2 - 4\lambda + 3$. The numbers $\lambda = 1$ and $\lambda = 3$ give $\det(A - \lambda I) = 0$. The (singular !) matrices are

$$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Note to instructor: You could explain that this is the reason determinants come before eigenvalues. Identify $\lambda = 1$ and $\lambda = 3$ as the eigenvalues of A.

23 $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ has det$(A) = 10$, $A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$, det$(A^2) = 100$, $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$ has det $\frac{1}{10}$. det$(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$ when $\lambda = \mathbf{2}$ or $\mathbf{5}$; those are eigenvalues.

24 Here $A = LU$ with $\det(L) = 1$ and $\det(U) = -6 =$ product of pivots, so also $\det(A) = -6$. $\det(U^{-1}L^{-1}) = -\frac{1}{6} = 1/\det(A)$ and $\det(U^{-1}L^{-1}A)$ is $\det I = 1$.

25 When the i, j entry is ij, row 2 = 2 times row 1 so $\det A = 0$.

26 When the ij entry is $i + j$, row 3 − row 2 = row2 − row 1 so A is singular: $\det A = 0$.

27 $\det A = abc$, $\det B = -abcd$, $\det C = a(b-a)(c-b)$ by doing elimination.

28 (a) *True*: $\det(AB) = \det(A)\det(B) = 0$ (b) *False*: A row exchange gives $-\det =$ product of pivots. (c) *False*: $A = 2I$ and $B = I$ have $A - B = I$ but the determinants have $2^n - 1 \neq 1$ (d) *True*: $\det(AB) = \det(A)\det(B) = \det(BA)$.

29 A is rectangular so $\det(A^{\mathrm{T}}A) \neq (\det A^{\mathrm{T}})(\det A)$: these determinants are not defined. In fact if A is tall and thin ($m > n$), then $\det(A^{\mathrm{T}}A)$ adds up $|\det B|^2$ where the B's are all the n by n submatrices of A.

30 Derivatives of $f = \ln(ad - bc)$:

$$\begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.$$

31 The Hilbert determinants are 1, 8×10^{-2}, 4.6×10^{-4}, 1.6×10^{-7}, 3.7×10^{-12}, 5.4×10^{-18}, 4.8×10^{-25}, 2.7×10^{-33}, 9.7×10^{-43}, 2.2×10^{-53}. Pivots are ratios of determinants so the 10th pivot is near 10^{-10}. The Hilbert matrix is numerically difficult (*ill-conditioned*). Please see the Figure 7.1 and Section 8.3.

32 Typical determinants of **rand**(n) are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. **randn**(n) with normal distribution gives $10^{31}, 10^{78}, 10^{186}$, **Inf** which means $\geq 2^{1024}$. MATLAB allows $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives **Inf**!

33 I now know that maximizing the determinant for $1, -1$ matrices is **Hadamard's problem** (1893): see Brenner in American Math. Monthly volume 79 (1972) 626-630. Neil Sloane's wonderful On-Line Encyclopedia of Integer Sequences (**research.att.com/~njas**) includes the solution for small n (and more references) when the problem is changed to $0, 1$ matrices. That sequence A003432 starts from $n = 0$ with 1, 1, 1, 2, 3, 5, 9. Then the $1, -1$ maximum for size n is 2^{n-1} times the $0, 1$ maximum for size $n - 1$ (so $(32)(5) = \mathbf{160}$ for $n = 6$ in sequence **A003433**).

To reduce the $1, -1$ problem from 6 by 6 to the $0, 1$ problem for 5 by 5, multiply the six rows by ± 1 to put $+1$ in column 1. Then subtract row 1 from rows 2 to 6 to get a 5 by 5 submatrix S with entries -2 and 0. Then divide S by -2.

Here is an advanced MATLAB code that finds a $1, -1$ matrix with largest $\det A = 48$ for $n = 5$:

```
n = 5; p = (n - 1)^2; A0 = ones(n); maxdet = 0;
for k = 0 : 2^p - 1
Asub = rem(floor(k .* 2.^(-p + 1 : 0)), 2); A = A0; A(2 : n, 2 : n) = 1 - 2*
reshape(Asub, n - 1, n - 1);
if abs(det(A)) > maxdet, maxdet = abs(det(A)); maxA = A;
end
end
```

Output: maxA =

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{matrix}$$

maxdet = 48.

34 Reduce B by row operations to $[\,\text{row } 3;\ \ \text{row } 2;\ \ \text{row } 1\,]$. Then $\det B = -6$ (odd permutation from the order of the rows in A).

Problem Set 5.2, page 266

1 $\det A = 1 + 18 + 12 - 9 - 4 - 6 = 12$, the rows of A are independent; $\det B = 0$, row 1 + row 2 = row 3 so the rows of B are linearly dependent; $\det C = -1$, so C has independent rows ($\det C$ has one term, an odd permutation).

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = -1$, independent but $\det D = 0$ because its submatrix B has dependent rows.

3 The problem suggests 3 ways to see that $\det A = 0$: All cofactors of row 1 are zero. A has rank ≤ 2. Each of the 6 terms in $\det A$ is zero. Notice also that column 2 has no pivot.

4 $a_{11}a_{23}a_{32}a_{44}$ gives -1, because the terms $a_{23}a_{32}$ have columns 2 and 3 in reverse order. $a_{14}a_{23}a_{32}a_{41}$ which has 2 row exchanges gives $+1$, $\det A = 1 - 1 = 0$. Using the same entries but now taken from B, $\det B = 2{\cdot}4{\cdot}4{\cdot}2 - 1{\cdot}4{\cdot}4{\cdot}1 = 64 - 16 = 48$.

5 Four zeros in the same row guarantee $\det = 0$ (and also four zeros in the same column). $A = I$ has 12 zeros (this is the maximum with $\det \neq 0$).

6 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms will be zeros (b) 15 terms must be zero. Effectively we are counting the permutations that make everyone move; $2, 3, 1$ and $3, 1, 2$ for $n = 3$ mean that the other 4 permutations take a term from the diagonal of A; so those terms are 0 when the diagonal is all zeros.

7 $5!/2 = 60$ permutation matrices (half of $5! = 120$ permutations) have $\det = +1$. Move row 5 of I to the top; then starting from $(5, 1, 2, 3, 4)$ elimination will do four row exchanges on P.

8 If $\det A \neq 0$, then certainly some term $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ in the big formula is not zero! Move rows $1, 2, \ldots, n$ into rows $\alpha, \beta, \ldots, \omega$. Then all these nonzero a's will be on the main diagonal.

9 The big formula has six terms all ± 1: say q are -1 and $6-q$ are 1. Then $\det A = -q+6-q =$ even (so $\det A = 5$ is impossible). Also $\det A = 6$ is impossible. All 3 even permutations like $a_{11}a_{22}a_{33}$ would have to give $+1$ (so an even number of -1's in the matrix). But all 3 odd permutations like $a_{11}a_{23}a_{32}$ would have to give -1 (so an odd number of -1's in the matrix). We can't have it both ways, so $\det A = 4$ is best possible and not hard to arrange: put -1's on the main diagonal.

10 The $4!/2 = 12$ even permutations are $(1,2,3,4), (2,1,4,3), (3,1,4,2), (4,3,2,1)$, and 8 P's with one number in place and even permutation of the other three numbers: examples are $1,3,4,2$ and $1,4,2,3$.

$\det(I + P_{\text{even}})$ is always 16 or 4 or 0 (16 comes from $I+I$).

11 $C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. $D = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$. $\det B = 1(0) + 2(42) + 3(-35) = -21$. Puzzle: $\det D = 441 = (-21)^2$. *Why* is $\det(\text{cofactor matrix}) = (\det\ \text{matrix})^{n-1}$?

12 $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore $A^{-1} = \frac{1}{4}C^{\mathrm{T}} = C^{\mathrm{T}}/\det A$.

13 (a) $C_1 = 0,\ C_2 = -1,\ C_3 = 0,\ C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row 1 then cofactors of column 1. Therefore $C_{10} = -C_8 = C_6 = -C_4 = C_2 = -1$.

14 For the matrices in Problem 13 to produce nonzeros in the big formula, we must choose 1's from column 2 then column 1, column 4 then column 3,and so on. Therefore n must be even to have $\det \neq 0$. The number of row exchanges is $n/2$ so the overall determinant is $C_n = (-1)^{n/2}$.

15 The $1,1$ cofactor of the n by n matrix is E_{n-1}. The $1,2$ cofactor has a single 1 in its first column, with cofactor E_{n-2}: sign gives $-E_{n-2}$. So $E_n = E_{n-1} - E_{n-2}$. Then E_1 to E_6 is $1,\ 0,\ -1,\ -1,\ 0,\ 1$ and this cycle of six will repeat: $E_{100} = E_4 = -1$.

16 The $1,1$ cofactor of the n by n matrix is F_{n-1}. The $1,2$ cofactor has a 1 in column 1, with cofactor F_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the $1,2$ entry to find $\boldsymbol{F_n = F_{n-1} + F_{n-2}}$. So these determinants are Fibonacci numbers.

17 Use cofactors along row 4 instead of row 1 (last row instead of first).

$$|B_4| = 2\det\begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} + \det\begin{bmatrix} 1 & -1 & \\ -1 & 2 & \\ & -1 & -1 \end{bmatrix} = 2|B_3| - \det\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

So $|B_4| = 2|B_3| - |B_2|$.

18 Rule 3 (linearity in row 1) gives $|B_n| = |A_n| - |A_{n-1}| = (n+1) - n = 1$.

19 Since x, x^2, x^3 are all in the same row, they never multiply each other in $\det V_4$. The determinant is zero at $x = a$ or b or c because of equal rows! So $\det V$ has factors $(x-a)(x-b)(x-c)$. Multiply by the cofactor V_3. The Vandermonde matrix $V_{ij} = (x_i)^{j-1}$ is for fitting a polynomial $p(\boldsymbol{x}) = \boldsymbol{b}$ at the points x_i. It has $\det V =$ product of all $x_k - x_m$ for $k > m$.

20 $G_2 = -1$, $G_3 = 2$, $G_4 = -3$, and $G_n = (-1)^{n-1}(n-1)$. One way to reach that G_n is to multiply the n eigenvalues $-1, -1, \ldots, -1, n-1$ of the matrix. Is there a good choice of row operations to produce this determinant G_n ?

21 $S_1 = 3, S_2 = 8, S_3 = 21$. The rule looks like every second number in Fibonacci's sequence $\ldots 3, 5, 8, 13, 21, 34, 55, \ldots$ so the guess is $S_4 = 55$. Following the solution to Problem 30 with 3's instead of 2's on the diagonal confirms $S_4 = 81+1-9-9-9 = 55$. Problem 32 directly proves $S_n = F_{2n+2}$.

22 Changing 3 to 2 in the corner reduces the determinant F_{2n+2} by 1 times the cofactor of that corner entry. This cofactor is the determinant of S_{n-1} (one size smaller) which is F_{2n}. Therefore changing 3 to 2 changes the determinant to $F_{2n+2} - F_{2n}$ which is Fibonacci's F_{2n+1}.

23 (a) If we choose an entry from B we must choose an entry from the zero block; result zero. This leaves entries from A times entries from D leading to $(\det A)(\det D)$

(b) and (c) Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. See #25.

24 (a) All the lower triangular blocks L_k have 1's on the diagonal and $\det = 1$. Then use $A_k = L_k U_k$ to find $\det U_k = \det A_k = 2, 6, -6$ for $k = 1, 2, 3$

(b) Equation (3) in this section gives the kth pivot as $\det A_k/\det A_{k-1}$. So $\det A_k = 5, 6, 7$ gives pivot $d_k = 5/1, 6/5, 7/6$.

25 Problem 23 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$. By the product rule this is $|AD - ACA^{-1}B|$. **If** $\boldsymbol{AC = CA}$ this is $|AD - CAA^{-1}B| = \mathbf{det}(\boldsymbol{AD - CB})$.

26 If A is a row and B is a column then $\det M = \det AB =$ dot product of A and B. If A is a column and B is a row then AB has rank 1 and $\det M = \det AB = 0$ (unless $m = n = 1$). This block matrix M is invertible when AB is invertible which certainly requires $m \leq n$.

27 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. Derivative with respect to $a_{11} =$ cofactor C_{11}.

28 Row $1 - 2$ row $2 +$ row $3 = 0$ so this matrix is singular and $\det A$ is zero.

29 There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs: $+\,(1,1)(2,2)(3,3)(4,4)\ +\ (1,2)(2,1)(3,4)(4,3)\ -\ (1,2)(2,1)(3,3)(4,4)\ -\ (1,1)(2,2)(3,4)(4,3)\ -\ (1,1)(2,3)(3,2)(4,4)$. Total -1.

30 The 5 products in solution 29 change to $16 + 1 - 4 - 4 - 4$ since A has 2's and -1's:

$$(2)(2)(2)(2) + (-1)(-1)(-1)(-1) - (-1)(-1)(2)(2) - (2)(2)(-1)(-1) - (2)(-1)(-1)(2) = \mathbf{5} = \boldsymbol{n} + \mathbf{1}.$$

31 $\det P = -1$ because the cofactor of $P_{14} = 1$ in row one has sign $(-1)^{1+4}$. The big formula for $\det P$ has only one term $(1 \cdot 1 \cdot 1 \cdot 1)$ with minus sign because three exchanges take $4, 1, 2, 3$ into $1, 2, 3, 4$; $\det(P^2) = (\det P)(\det P) = +1$ so

$$\det \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ is } \textit{not right}.$$

32 The problem is to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using Fibonacci's rule:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = 2F_{2n} + (F_{2n} - F_{2n-2}) = 3F_{2n} - F_{2n-2}.$$

33 The difference from 20 to 19 multiplies its 3 by 3 cofactor $= 1$: then det drops by 1.

34 (a) The last three rows must be dependent because only 2 columns are nonzero

(b) In each of the 120 terms: Choices from the last 3 rows must use 3 different columns; at least one of those choices will be zero.

35 Subtracting 1 from the n, n entry subtracts its cofactor C_{nn} from the determinant. That cofactor is $C_{nn} = 1$ (smaller Pascal matrix). Subtracting 1 from 1 leaves 0.

Problem Set 5.3, page 283

1 (a) $|A| = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 3,\ |B_1| = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 6,\ |B_2| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ so $x_1 = -6/3 = -2$ and $x_2 = 3/3 = 1$ (b) $|A| = 4, |B_1| = 3, |B_2| = 2, |B_3| = 1$. Therefore $x_1 = 3/4$ and $x_2 = -1/2$ and $x_3 = 1/4$.

2 (a) $y = \begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix} / \begin{vmatrix} a & b \\ c & d \end{vmatrix} = c/(ad - bc)$ (b) $y = \det B_2 / \det A = (fg - id)/D$. That is because B_2 with $(1, 0, 0)$ in column 2 has $\det B_2 = fg - id$.

3 (a) $x_1 = 3/0$ and $x_2 = -2/0$: *no solution* (b) $x_1 = x_2 = \mathbf{0}/\mathbf{0}$: *undetermined*.

4 (a) $x_1 = \det\big([\,\boldsymbol{b}\ \ \boldsymbol{a}_2\ \ \boldsymbol{a}_3\,]\big)/\det A$, if $\det A \neq 0$. This is $|B_1|/|A|$.

(b) The determinant is linear in its first column so $|x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3\boldsymbol{a}_2\boldsymbol{a}_3|$ splits into $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3| + x_2|\boldsymbol{a}_2\ \boldsymbol{a}_2\ \boldsymbol{a}_3| + x_3|\boldsymbol{a}_3\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$. The last two determinants are zero because of repeated columns, leaving $x_1|\boldsymbol{a}_1\ \boldsymbol{a}_2\ \boldsymbol{a}_3|$ which is $x_1 \det A$.

5 If the first column in A is also the right side b then $\det A = \det B_1$. Both B_2 and B_3 are singular since a column is repeated. Therefore $x_1 = |B_1|/|A| = 1$ and $x_2 = x_3 = 0$.

6 (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{7}{3} & 1 \end{bmatrix}$ (b) $\frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. An invertible symmetric matrix has a symmetric inverse.

7 If all cofactors $= 0$ then A^{-1} would be the zero matrix if it existed; cannot exist. (And also, the cofactor formula gives $\det A = 0$.) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has no zero cofactors but it is not invertible.

8 $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. This is $(\det A)I$ and $\det A = 3$. The $1, 3$ cofactor of A is 0. Then $C_{31} = 4$ or 100: no change.

9 If we know the cofactors and $\det A = 1$, then $C^{\mathrm{T}} = A^{-1}$ and also $\det A^{-1} = 1$. Now A is the inverse of C^{T}, so A can be found from the cofactor matrix for C.

10 Take the determinant of $AC^{\mathrm{T}} = (\det A)I$. The left side gives $\det AC^{\mathrm{T}} = (\det A)(\det C)$ while the right side gives $(\det A)^n$. Divide by $\det A$ to reach $\det C = (\det A)^{n-1}$.

11 The cofactors of A are integers. Division by $\det A = \pm 1$ gives integer entries in A^{-1}.

12 Both $\det A$ and $\det A^{-1}$ are integers since the matrices contain only integers. But $\det A^{-1} = 1/\det A$ so $\det A$ must be 1 or -1.

13 $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ has cofactor matrix $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{bmatrix}$ and $A^{-1} = \frac{1}{5}C^{\mathrm{T}}$.

14 (a) Lower triangular L has cofactors $C_{21} = C_{31} = C_{32} = 0$ (b) $C_{12} = C_{21}$, $C_{31} = C_{13}, C_{32} = C_{23}$ make S^{-1} symmetric. (c) Orthogonal Q has cofactor matrix $C = (\det Q)(Q^{-1})^{\mathrm{T}} = \pm Q$ *also orthogonal*. Note $\det Q = 1$ or -1.

15 For $n = 5$, C contains *25* cofactors and each 4 by 4 cofactor has *24* terms. Each term needs *3* multiplications: total 1800 multiplications vs.125 for Gauss-Jordan.

16 (a) Area $\begin{vmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{4} \end{vmatrix} = 10$ (b) and (c) Area $10/2 = 5$, these triangles are half of the parallelogram in (a).

17 Volume$=\begin{vmatrix} \mathbf{3} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{3} \end{vmatrix}=\mathbf{20}$. Area of faces $=$ length of cross product $= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \mathbf{3} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \end{vmatrix} =$ $-2\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k}$ length$=\mathbf{6\sqrt{2}}$

18 (a) Area $\frac{1}{2}\begin{vmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{4} & \mathbf{1} \\ \mathbf{0} & \mathbf{5} & \mathbf{1} \end{vmatrix} = 5$ (b) $5 +$ new triangle area $\frac{1}{2}\begin{vmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{5} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{vmatrix} = 5 + 7 = 12$.

19 $\begin{vmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} \end{vmatrix} = 4 = \begin{vmatrix} \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} \end{vmatrix}$ because the transpose has the same determinant. See $\#22$.

20 The edges of the hypercube have length $\sqrt{1+1+1+1} = 2$. The volume $\det H$ is $2^4 = \mathbf{16}$. ($H/2$ has orthonormal columns. Then $\det(H/2) = 1$ leads again to $\det H = 16$ in 4 dimensions.)

21 The maximum volume $L_1L_2L_3L_4$ is reached when the edges are orthogonal in $\mathbf{R}^4$. With entries 1 and -1 all lengths are $\sqrt{4} = 2$. The maximum determinant is $2^4 = 16$, achieved in Problem 20. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can't be achieved by ± 1. $\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

22 This question is still waiting for a solution! An 18.06 student showed me how to transform the parallelogram for A to the parallelogram for A^{T}, without changing its area. (Edges slide along themselves, so no change in baselength or height or area.)

23 $A^{\mathrm{T}}A = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}} \\ \boldsymbol{c}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} & 0 & 0 \\ 0 & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} & 0 \\ 0 & 0 & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} \end{bmatrix}$ has $\begin{array}{lcl} \det A^{\mathrm{T}}A & = & (\|a\|\|b\|\|c\|)^2 \\ \det A & = & \pm\|a\|\|b\|\|c\| \end{array}$

24 The box has height 4 and volume $= \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = 4$. $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$ and $(\boldsymbol{k} \cdot \boldsymbol{w}) = 4$.

25 The n-dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. Coefficients from $(2+x)^n$ in Worked Example **2.4A**. Cube from $2I$ has volume 2^n.

26 The pyramid has volume $\frac{1}{6}$. The 4-dimensional pyramid has volume $\frac{1}{24}$ (and $\frac{1}{n!}$ in $\mathbf{R}^n$)

27 $x = r\cos\theta, y = r\sin\theta$ give $J = r$. This is the r in polar area $r\,dr\,d\theta$. The columns are orthogonal and their lengths are 1 and r.

28 $J = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & \theta \end{vmatrix} = \rho^2\sin\varphi$. This Jacobian is needed for triple integrals inside spheres. Those integrals have $\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

29 From x, y to r, θ: $\begin{vmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{vmatrix} = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ (-\sin\theta)/r & (\cos\theta)/r \end{vmatrix}$ $= \dfrac{1}{r} = \dfrac{1}{\text{Jacobian in } \mathbf{27}}$. The surprise was that $\frac{dr}{dx}$ and $\frac{dx}{dr}$ are both $\frac{x}{r}$.

30 The triangle with corners $(0,0), (6,0), (1,4)$ has area $(6)(4)/2 = 12$. Rotated by $\theta = 60^\circ$ the area is *unchanged*. The determinant of the rotation matrix is

$$J = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \begin{vmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{vmatrix} = 1.$$

31 Base area $||\boldsymbol{u} \times \boldsymbol{v}|| = 10$, height $||\boldsymbol{w}|| \cos\theta = 2$, volume $(10)(2) = 20$.

32 The volume of the box is $\det \begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 0 \\ 1 & 2 & 2 \end{bmatrix} = 20$, agreeing with Problem 31.

33 $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$. This is $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$.

34 $(\boldsymbol{w} \times \boldsymbol{u}) \cdot \boldsymbol{v} = (\boldsymbol{v} \times \boldsymbol{w}) \cdot \boldsymbol{u} = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$: *Even permutation* of $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ keeps the same determinant. Odd permutations like $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{v}$ will reverse the sign.

35 $S = (2, 1, -1)$, area $\|PQ \times PS\| = \|(-2, -2, -1)\| = \sqrt{2^2 + 2^2 + 1^2} = 3$. The other four corners of the box can be $(0,0,0)$, $(0,0,2)$, $(1,2,2)$, $(1,1,0)$. The volume of the tilted box with edges along P, Q, and R is $|\det| = 1$.

36 If $(1,1,0)$, $(1,2,1)$, (x,y,z) are in a plane the volume is $\det \begin{bmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = x - y + z = 0$. The "box" with those edges is flattened to zero height.

37 $\det \begin{bmatrix} x & y & z \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} = 7x - 5y + z$ will be *zero* when (x, y, z) is a combination of $(2, 3, 1)$ and $(1, 2, 3)$. The plane containing those two vectors has equation $7x - 5y + z = 0$. Volume = zero because the 3 box edges out from $(0,0,0)$ lie in a plane.

38 Doubling each row multiplies the volume by 2^n. Then $2\det A = \det(2A)$ only if $n=1$.

39 $AC^{\mathrm{T}} = (\det A)I$ gives $(\det A)(\det C) = (\det A)^n$. Then $\det A = (\det C)^{1/3}$ with $n = 4$. With $\det A^{-1} = 1/\det A$, construct A^{-1} using the cofactors. *Invert to find* A.

40 The cofactor formula adds 1 by 1 determinants (which are just entries) *times* their cofactors of size $n-1$. Jacobi discovered that this formula can be generalized. For $n = 5$, Jacobi multiplied each 2 by 2 determinant from rows 1-2 (with columns $a < b$) times a 3 by 3 determinant from rows 3-5 (using the remaining columns $c < d < e$).

The key question is $+$ or $-$ sign (as for cofactors). The product is given a $+$ sign when a, b, c, d, e is an even permutation of 1, 2, 3, 4, 5. This gives the correct determinant $+1$ for that permutation matrix. More than that, all other P that permute a, b and separately c, d, e will come out with the correct sign when the 2 by 2 determinant for columns a, b multiplies the 3 by 3 determinant for columns c, d, e.

41 The Cauchy-Binet formula gives the determinant of a square matrix AB (and AA^{T} in particular) when the factors A, B are rectangular. For (2 by 3) times (3 by 2) there are 3 products of 2 by 2 determinants from A and B (printed in boldface):

$$\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & c \\ \boldsymbol{d} & \boldsymbol{e} & f \end{bmatrix} \begin{bmatrix} \boldsymbol{g} & \boldsymbol{j} \\ \boldsymbol{h} & \boldsymbol{k} \\ i & \ell \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{a} & b & \boldsymbol{c} \\ \boldsymbol{d} & e & \boldsymbol{f} \end{bmatrix} \begin{bmatrix} \boldsymbol{g} & \boldsymbol{j} \\ h & k \\ \boldsymbol{i} & \boldsymbol{\ell} \end{bmatrix} \qquad \begin{bmatrix} a & \boldsymbol{b} & \boldsymbol{c} \\ d & \boldsymbol{e} & \boldsymbol{f} \end{bmatrix} \begin{bmatrix} g & j \\ \boldsymbol{h} & \boldsymbol{k} \\ \boldsymbol{i} & \boldsymbol{\ell} \end{bmatrix}$$

$$\textit{Check} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \qquad AB = \begin{bmatrix} 14 & 30 \\ 30 & 66 \end{bmatrix}$$

Cauchy-Binet: $(4-2)(4-2) + (7-3)(7-3) + (14-12)(14-12) = \mathbf{24}$

det of AB : $(14)(66) - (30)(30) = \mathbf{24}$

42 A 5 by 5 tridiagonal matrix has cofactor $C_{11} = 4$ by 4 tridiagonal matrix. Cofactor C_{12} has only one nonzero at the top of column 1. That nonzero multiplies its 3 by 3 cofactor which is tridiagonal. So $\det A = a_{11}C_{11} + a_{12}C_{12} =$ tridiagonal determinants of sizes 4 and 3. The number F_n of nonzero terms in $\det A$ follows Fibonacci's rule $F_n = F_{n-1} + F_{n-2}$.