INTRODUCTION
TO
LINEAR
ALGEBRA
Fifth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 1.1, page 8

1 The combinations give (a) a line in \( \mathbb{R}^3 \) (b) a plane in \( \mathbb{R}^3 \) (c) all of \( \mathbb{R}^3 \).

2 \( \mathbf{v} + \mathbf{w} = (2, 3) \) and \( \mathbf{v} - \mathbf{w} = (6, -1) \) will be the diagonals of the parallelogram with \( \mathbf{v} \) and \( \mathbf{w} \) as two sides going out from \((0, 0)\).

3 This problem gives the diagonals \( \mathbf{v} + \mathbf{w} \) and \( \mathbf{v} - \mathbf{w} \) of the parallelogram and asks for the sides: The opposite of Problem 2. In this example \( \mathbf{v} = (3, 3) \) and \( \mathbf{w} = (2, -2) \).

4 \( 3\mathbf{v} + \mathbf{w} = (7, 5) \) and \( c\mathbf{v} + d\mathbf{w} = (2c + d, c + 2d) \).

5 \( \mathbf{u} + \mathbf{v} = (-2, 3, 1) \) and \( \mathbf{u} + \mathbf{v} + \mathbf{w} = (0, 0, 0) \) and \( 2\mathbf{u} + 2\mathbf{v} + \mathbf{w} = (\text{add first answers}) = (-2, 3, 1) \). The vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) are in the same plane because a combination gives \((0, 0, 0)\). Stated another way: \( \mathbf{u} = -\mathbf{v} - \mathbf{w} \) is in the plane of \( \mathbf{v} \) and \( \mathbf{w} \).

6 The components of every \( c\mathbf{v} + d\mathbf{w} \) add to zero because the components of \( \mathbf{v} \) and of \( \mathbf{w} \) add to zero. \( c = 3 \) and \( d = 9 \) give \((3, 3, -6)\). There is no solution to \( c\mathbf{v} + d\mathbf{w} = (3, 3, 6) \) because \( 3 + 3 + 6 \) is not zero.

7 The nine combinations \( c(2, 1) + d(0, 1) \) with \( c = 0, 1, 2 \) and \( d = (0, 1, 2) \) will lie on a lattice. If we took all whole numbers \( c \) and \( d \), the lattice would lie over the whole plane.

8 The other diagonal is \( \mathbf{v} - \mathbf{w} \) (or else \( \mathbf{w} - \mathbf{v} \)). Adding diagonals gives \( 2\mathbf{v} \) (or \( 2\mathbf{w} \)).

9 The fourth corner can be \((4, 4)\) or \((4, 0)\) or \((-2, 2)\). Three possible parallelograms!

10 \( \mathbf{i} - \mathbf{j} = (1, 1, 0) \) is in the base \((x,y)\) plane. \( \mathbf{i} + \mathbf{j} + \mathbf{k} = (1, 1, 1) \) is the opposite corner from \((0, 0, 0)\). Points in the cube have \( 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \).

11 Four more corners \((1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\). The center point is \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\).

Centers of faces are \((\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)\) and \((0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})\) and \((\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})\).

12 The combinations of \( \mathbf{i} = (1, 0, 0) \) and \( \mathbf{i} + \mathbf{j} = (1, 1, 0) \) fill the \( xy \) plane in \( xyz \) space.

13 Sum = zero vector. Sum = \(-2:00 \) vector = \( 8:00 \) vector. \( 2:00 \) is \( 30^\circ \) from horizontal = \((\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\sqrt{3}/2, 1/2)\).

14 Moving the origin to \( 6:00 \) adds \( \mathbf{j} = (0, 1) \) to every vector. So the sum of twelve vectors changes from \( 0 \) to \( 12\mathbf{j} = (0, 12) \).
Solutions to Exercises

15. The point \( \frac{3}{4}v + \frac{1}{4}w \) is three-fourths of the way to \( v \) starting from \( w \). The vector \( \frac{1}{4}v + \frac{1}{4}w \) is halfway to \( u = \frac{1}{2}v + \frac{1}{2}w \). The vector \( v + w \) is \( 2u \) (the far corner of the parallelogram).

16. All combinations with \( c + d = 1 \) are on the line that passes through \( v \) and \( w \). The point \( V = -v + 2w \) is on that line but it is beyond \( w \).

17. All vectors \( cv + dw \) are on the line passing through \((0, 0)\) and \( u = \frac{1}{2}v + \frac{1}{2}w \). That line continues out beyond \( v + w \) and back beyond \((0, 0)\). With \( c \geq 0 \), half of this line is removed, leaving a ray that starts at \((0, 0)\).

18. The combinations \( cv + dw \) with \( 0 \leq c \leq 1 \) and \( 0 \leq d \leq 1 \) fill the parallelogram with sides \( v \) and \( w \). For example, if \( v = (1, 0) \) and \( w = (0, 1) \) then \( cv + dw \) fills the unit square. But when \( v = (a, 0) \) and \( w = (b, 0) \) these combinations only fill a segment of a line.

19. With \( c \geq 0 \) and \( d \geq 0 \) we get the infinite “cone” or “wedge” between \( v \) and \( w \). For example, if \( v = (1, 0) \) and \( w = (0, 1) \), then the cone is the whole quadrant \( x \geq 0, y \geq 0 \). Question: What if \( w = -v \)? The cone opens to a half-space. But the combinations of \( v = (1, 0) \) and \( w = (-1, 0) \) only fill a line.

20. (a) \( \frac{1}{2}u + \frac{1}{3}v + \frac{1}{3}w \) is the center of the triangle between \( u, v \) and \( w \); \( \frac{1}{2}u + \frac{1}{2}w \) lies between \( u \) and \( w \)  
(b) To fill the triangle keep \( c \geq 0, d \geq 0, e \geq 0 \), and \( c + d + e = 1 \).

21. The sum is \((v - u) + (w - v) + (u - w) = \text{zero vector}\). Those three sides of a triangle are in the same plane!

22. The vector \( \frac{1}{2}(u + v + w) \) is outside the pyramid because \( c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1 \).

23. All vectors are combinations of \( u, v, w \) as drawn (not in the same plane). Start by seeing that \( cu + dv \) fills a plane, then adding \( ew \) fills all of \( \mathbb{R}^3 \).

24. The combinations of \( u \) and \( v \) fill one plane. The combinations of \( v \) and \( w \) fill another plane. Those planes meet in a line: only the vectors \( cv \) are in both planes.

25. (a) For a line, choose \( u = v = w = \text{any nonzero vector} \)  
(b) For a plane, choose \( u \) and \( v \) in different directions. A combination like \( w = u + v \) is in the same plane.
26 Two equations come from the two components: \( c + 3d = 14 \) and \( 2c + d = 8 \). The solution is \( c = 2 \) and \( d = 4 \). Then \( 2(1, 2) + 4(3, 1) = (14, 8) \).

27 A four-dimensional cube has \( 2^4 = 16 \) corners and \( 2 \times 4 = 8 \) three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example 2.4 A.

28 There are 6 unknown numbers \( v_1, v_2, v_3, w_1, w_2, w_3 \). The six equations come from the components of \( v + w = (4, 5, 6) \) and \( v - w = (2, 5, 8) \). Add to find \( 2v = (6, 10, 14) \) so \( v = (3, 5, 7) \) and \( w = (1, 0, -1) \).

29 Two combinations out of infinitely many that produce \( b = (0, 1) \) are \(-2u + v\) and \( \frac{1}{2}w - \frac{1}{2}v\). No, three vectors \( u, v, w \) in the \( x-y \) plane could fail to produce \( b \) if all three lie on a line that does not contain \( b \). Yes, if one combination produces \( b \) then two (and infinitely many) combinations will produce \( b \). This is true even if \( u = 0 \); the combinations can have different \( cu \).

30 The combinations of \( v \) and \( w \) fill the plane unless \( v \) and \( w \) lie on the same line through \( (0, 0) \). Four vectors whose combinations fill 4-dimensional space: one example is the “standard basis” \( (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), \) and \( (0, 0, 0, 1) \).

31 The equations \( cu + dv + ew = b \) are

\[
\begin{align*}
2c & -d = 1 \\
-c + 2d & -e = 0 \\
-d + 2e & = 0
\end{align*}
\]

So \( d = 2e \) then \( c = 3e \) and \( d = 2/4 \) then \( 4e = 1 \) and \( e = 1/4 \).

Problem Set 1.2, page 18

1 \( u \cdot v = -2.4 + 2.4 = 0, u \cdot w = -1.6 + 1.6 = 0, w \cdot v = 4 - 6 = -2 = v \cdot w. \)

2 \( \|u\| = 1 \) and \( \|v\| = 5 \) and \( \|w\| = \sqrt{5} \). Then \( |u \cdot v| = 0 < (1)(5) \) and \( |v \cdot w| = 10 < 5\sqrt{5} \), confirming the Schwarz inequality.
3 Unit vectors \( \mathbf{v}/||\mathbf{v}|| = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (0.8, 0.6) \). The vectors \( \mathbf{w}, (2, -1), \) and \(-\mathbf{w}\) make 0°, 90°, 180° angles with \( \mathbf{w} \) and \( \mathbf{v}/||\mathbf{w}|| = (1/\sqrt{5}, 2/\sqrt{5}) \). The cosine of \( \theta \) is \( \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}|| \cdot ||\mathbf{w}||} = 10/5\sqrt{5} \).

4 (a) \( \mathbf{v} \cdot (-\mathbf{v}) = -1 \) (b) \( (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{w} = 1+0-0 = 1 = 0 \) so \( \theta = 90° \) (notice \( \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \) ) (c) \( (\mathbf{v} - 2\mathbf{w}) \cdot (\mathbf{v} + 2\mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - 4\mathbf{w} \cdot \mathbf{w} = 1 - 4 = -3 \).

5 \( \mathbf{u}_1 = \mathbf{v}/||\mathbf{v}|| = (1, 3)/\sqrt{10} \) and \( \mathbf{u}_2 = \mathbf{w}/||\mathbf{w}|| = (2, 1, 2)/3 \). \( \mathbf{U}_1 = (3, -1)/\sqrt{10} \) is perpendicular to \( \mathbf{u}_1 \) (and so is \( (-3, 1)/\sqrt{10} \)). \( \mathbf{U}_2 \) could be \( (1, -2, 0)/\sqrt{5} \). There is a whole plane of vectors perpendicular to \( \mathbf{u}_2 \), and a whole circle of unit vectors in that plane.

6 All vectors \( \mathbf{w} = (c, 2c) \) are perpendicular to \( \mathbf{v} \). They lie on a line. All vectors \((x, y, z)\) with \( x + y + z = 0 \) lie on a plane. All vectors perpendicular to \((1, 1, 1)\) and \((1, 2, 3)\) lie on a line in 3-dimensional space.

7 (a) \( \cos \theta = \mathbf{v} \cdot \mathbf{w}/||\mathbf{v}|| ||\mathbf{w}|| = 1/(2)(1) \) so \( \theta = 60° \) or \( \pi/3 \) radians (b) \( \cos \theta = 0 \) so \( \theta = 90° \) or \( \pi/2 \) radians (c) \( \cos \theta = 2/(2)(2) = 1/2 \) so \( \theta = 60° \) or \( \pi/3 \) (d) \( \cos \theta = -1/\sqrt{2} \) so \( \theta = 135° \) or \( 3\pi/4 \).

8 (a) False: \( \mathbf{v} \) and \( \mathbf{w} \) are any vectors in the plane perpendicular to \( \mathbf{u} \) (b) True: \( \mathbf{u} \cdot (\mathbf{v} + 2\mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + 2\mathbf{u} \cdot \mathbf{w} = 0 \) (c) True, \( ||\mathbf{u} - \mathbf{v}||^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) \) splits into \( \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = 2 \) when \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = 0 \).

9 If \( v_2w_2/v_1w_1 = -1 \) then \( v_2w_2 = -v_1w_1 \) or \( v_1w_1 + v_2w_2 = v \cdot w = 0 \): perpendicular!

The vectors \((1, 4)\) and \((1, -\frac{3}{2})\) are perpendicular.

10 Slopes 2/1 and \(-1/2\) multiply to give \(-1\): then \( \mathbf{v} \cdot \mathbf{w} = 0 \) and the vectors (the directions) are perpendicular.

11 \( \mathbf{v} \cdot \mathbf{w} < 0 \) means angle > 90°; these \( \mathbf{w} \)'s fill half of 3-dimensional space.

12 \((1, 1)\) perpendicular to \((1, 5)\) if \((1, 1) \cdot (1, 5) - c(1, 1) \cdot (1, 1) = 6 - 2c = 0 \) or \( c = 3 \); \( \mathbf{v} \cdot (\mathbf{w} - c\mathbf{v}) = 0 \) if \( c = \mathbf{v} \cdot \mathbf{w}/\mathbf{v} \cdot \mathbf{v} \). Subtracting \( c\mathbf{v} \) is the key to constructing a perpendicular vector.
13 The plane perpendicular to \((1, 0, 1)\) contains all vectors \((c, d, -c)\). In that plane, \(v = (1, 0, -1)\) and \(w = (0, 1, 0)\) are perpendicular.

14 One possibility among many: \(u = (1, -1, 0, 0), v = (0, 0, 1, -1), w = (1, 1, -1, -1)\) and \((1, 1, 1, 1)\) are perpendicular to each other. “We can rotate those \(u, v, w\) in their 3D hyperplane and they will stay perpendicular.”

15 \(\frac{1}{2}(x + y) = (2 + 8)/2 = 5\) and \(5 > 4\); \(\cos \theta = 2 \sqrt{16}/\sqrt{10} \sqrt{10} = 8/10\).

16 \(\|v\|^2 = 1 + 1 + \cdots + 1 = 9\) so \(\|v\| = 3; u = v/3 = (\frac{1}{3}, \ldots, \frac{1}{3})\) is a unit vector in 9D; \(w = (1, -1, 0, \ldots, 0)/\sqrt{2}\) is a unit vector in the 8D hyperplane perpendicular to \(v\).

17 \(\cos \alpha = 1/\sqrt{2}, \cos \beta = 0, \cos \gamma = -1/\sqrt{2}\). For any vector \(v = (v_1, v_2, v_3)\) the cosines with \((1, 0, 0)\) and \((0, 0, 1)\) are \(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = (v_1^2 + v_2^2 + v_3^2)/\|v\|^2 = 1\).

18 \(\|v\|^2 = 4^2 + 2^2 = 20\) and \(\|w\|^2 = (-1)^2 + 2^2 = 5\). Pythagoras is \(\|(3, 4, 0)\|^2 = 25 = 20 + 5\) for the length of the hypotenuse \(v + w = (3, 4)\).

19 Start from the rules (1), (2), (3) for \(v \cdot w = w \cdot v\) and \(u \cdot (v + w)\) and \((cv) \cdot w\). Use rule (2) for \((v + w) \cdot (v + w) = (v + w) \cdot v + (v + w) \cdot w\). By rule (1) this is \(v \cdot (v + w) + w \cdot (v + w)\). Rule (2) again gives \(v \cdot v + v \cdot w + w \cdot v + w \cdot w = v \cdot v + 2v \cdot w + w \cdot w\). Notice \(v \cdot w = w \cdot v\)!

20 We know that \((v - w) \cdot (v - w) = v \cdot v - 2v \cdot w + w \cdot w\). The Law of Cosines writes \(\|v||w||\cos \theta\) for \(v \cdot w\). Here \(\theta\) is the angle between \(v\) and \(w\). When \(\theta < 90^\circ\) this \(v \cdot w\) is positive, so in this case \(v \cdot v + w \cdot w\) is larger than \(\|v - w\|^2\).

Pythagoras changes from equality \(a^2 + b^2 = c^2\) to inequality when \(\theta < 90^\circ\) or \(\theta > 90^\circ\).

21 \(2v \cdot w \leq 2||v||\|w\|\) leads to \(\|v + w\|^2 = v \cdot v + 2v \cdot w + w \cdot w \leq ||v||^2 + 2||v||\|w\| + \|w\|^2\). This is \((||v|| + \|w\||)^2\). Taking square roots gives \(\|v + w\| \leq ||v|| + \|w\||\).

22 \(v_1^2 w_1^2 + 2v_1 w_1 v_2 w_2 + v_2^2 w_2^2 \leq v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2\) is true (cancel 4 terms) because the difference is \(v_1^2 w_2^2 + v_2^2 w_1^2 \leq 2v_1 v_2 w_1 w_2\) which is \((v_1 w_2 - v_2 w_1)^2 \geq 0\).

23 \(\cos \beta = w_1/\|w\|\) and \(\sin \beta = w_2/\|w\|\). Then \(\cos(\beta - \alpha) = \cos \beta \cos \alpha + \sin \beta \sin \alpha = v_1 w_1/\|v||w\| + v_2 w_2/\|v||w\| = v \cdot w/\|v||w\|\). This is \(\cos \theta\) because \(\beta - \alpha = \theta\).
Example 6 gives \(|u_1||U_1| \leq \frac{1}{2}(u_1^2 + U_1^2)\) and \(|u_2||U_2| \leq \frac{1}{2}(u_2^2 + U_2^2)\). The whole line becomes \(0.96 \leq (0.6)(0.8) + (0.8)(0.6) \leq \frac{1}{2}(0.6^2 + 0.8^2) + \frac{1}{2}(0.8^2 + 0.6^2) = 1\). True: \(0.96 < 1\).

25 The cosine of \(\theta\) is \(x/\sqrt{x^2 + y^2}\), near side over hypotenuse. Then \(\cos \theta\) is not greater than 1: \(x^2/(x^2 + y^2) \leq 1\).

26 The vectors \(w = (x, y)\) with \((1, 2) \cdot w = x + 2y = 5\) lie on a line in the \(xy\) plane. The shortest \(w\) on that line is \((1, 2)\). (The Schwarz inequality \(\|v\| \geq v \cdot w/\|w\| = \sqrt{5}\) is an equality when \(\cos \theta = 0\) and \(w = (1, 2)\) and \(\|w\| = \sqrt{5}\).)

27 The length \(\|v - w\|\) is between 2 and 8 (triangle inequality when \(\|v\| = 5\) and \(\|w\| = 3\). The dot product \(v \cdot w\) is between \(-15\) and 15 by the Schwarz inequality.

28 Three vectors in the plane could make angles greater than 90° with each other: for example \((1, 0), (-1, 4), (-1, -4)\). Four vectors could not do this (360° total angle). How many can do this in \(R^3\) or \(R^n\)? Ben Harris and Greg Marks showed me that the answer is \(n + 1\). The vectors from the center of a regular simplex in \(R^n\) to its \(n + 1\) vertices all have negative dot products. If \(n + 2\) vectors in \(R^n\) had negative dot products, project them onto the plane orthogonal to the last one. Now you have \(n + 1\) vectors in \(R^{n-1}\) with negative dot products. Keep going to 4 vectors in \(R^2\): no way!

29 For a specific example, pick \(v = (1, 2, -3)\) and then \(w = (-3, 1, 2)\). In this example \(\cos \theta = v \cdot w/\|v\|\|w\| = -7/\sqrt{14}\sqrt{14} = -1/2\) and \(\theta = 120^\circ\). This always happens when \(x + y + z = 0\):

\[
\begin{align*}
v \cdot w &= xz + xy + yz = \frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2) \\
&= -\frac{1}{2} \cdot \|v\|\|w\|. Then \cos \theta = \frac{1}{2}.
\end{align*}
\]

30 Wikipedia gives this proof of geometric mean \(G = \sqrt[3]{xyz} \leq \text{arithmetic mean } A = (x + y + z)/3\). First there is equality in case \(x = y = z\). Otherwise \(A\) is somewhere between the three positive numbers, say for example \(z < A < y\).

Use the known inequality \(g \leq a\) for the two positive numbers \(x\) and \(y + z - A\). Their mean \(a = \frac{1}{2}(x + y + z - A)\) is \(\frac{1}{2}(3A - A) = \text{same as } A\). So \(a \geq g\) says that
\[ A^3 \geq g^2 A = x(y + z - A)A. \] But \((y + z - A)A = (y - A)(A - z) + yz > yz.\n
Substitute to find \(A^3 > xyz = G^3\) as we wanted to prove. Not easy!

There are many proofs of \(G = (x_1x_2 \cdots x_n)^{1/n} \leq A = (x_1 + x_2 + \cdots + x_n)/n.\) In calculus you are maximizing \(G\) on the plane \(x_1 + x_2 + \cdots + x_n = n.\) The maximum occurs when all \(x\)'s are equal.

31 The columns of the 4 by 4 “Hadamard matrix” (times \(\frac{1}{2}\)) are perpendicular unit vectors:
\[
\frac{1}{2} H = \frac{1}{2} \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}.
\]

32 The commands \(V = \text{randn}(3, 30); D = \text{sqrt} \left( \text{diag}(V' \ast V) \right); U = V \backslash D;\) will give 30 random unit vectors in the columns of \(U.\) Then \(u' \ast U\) is a row matrix of 30 dot products whose average absolute value may be close to \(2/\pi.\)

Problem Set 1.3, page 29

1 \(2s_1 + 3s_2 + 4s_3 = (2, 5, 9).\) The same vector \(b\) comes from \(S\) times \(x = (2, 3, 4):\)
\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
2 \\
3 \\
4
\end{bmatrix}
= \begin{bmatrix}
(\text{row 1}) \cdot x \\
(\text{row 2}) \cdot x \\
(\text{row 2}) \cdot x
\end{bmatrix}
= \begin{bmatrix}
2 \\
5 \\
9
\end{bmatrix}.
\]

2 The solutions are \(y_1 = 1, y_2 = 0, y_3 = 0\) (right side = column 1) and \(y_1 = 1, y_2 = 3, y_3 = 5.\) That second example illustrates that the first \(n\) odd numbers add to \(n^2.\)

3 \(y_1 + y_2 = B_2\) gives \(y_2 = -B_1 + B_2 = \begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0
\end{bmatrix} \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}\)
\(y_1 + y_2 + y_3 = B_3\) \(y_3 = -B_2 + B_3 = \begin{bmatrix}
0 & -1 & 1
\end{bmatrix} \begin{bmatrix}
B_3
\end{bmatrix}\)
The inverse of $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Independent columns in $A$ and $S$!

4 The combination $0w_1 + 0w_2 + 0w_3$ always gives the zero vector, but this problem looks for other zero combinations (then the vectors are dependent, they lie in a plane): $w_2 = (w_1 + w_3)/2$ so one combination that gives zero is $1/2w_1 - w_2 + 1/2w_3 = 0$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be dependent: $r_2 = 1/2(r_1 + r_3)$.

The column and row combinations that produce 0 are the same: this is unusual. Two solutions to $y_1r_1 + y_2r_2 + y_3r_3 = 0$ are $(Y_1, Y_2, Y_3) = (1, -2, 1)$ and $(2, -4, 2)$.

6 $c = 3 \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ has column 3 = column 1 - column 2

$c = -1 \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has column 3 = - column 1 + column 2

$c = 0 \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$ has column 3 = 3 (column 1) - column 2

7 All three rows are perpendicular to the solution $x$ (the three equations $r_1 \cdot x = 0$ and $r_2 \cdot x = 0$ and $r_3 \cdot x = 0$ tell us this). Then the whole plane of the rows is perpendicular to $x$ (the plane is also perpendicular to all multiples $cx$).

$$
\begin{align*}
x_1 - 0 &= b_1 & x_1 &= b_1 \\
x_2 - x_1 &= b_2 & x_2 &= b_1 + b_2 \\
x_3 - x_2 &= b_3 & x_3 &= b_1 + b_2 + b_3 \\
x_4 - x_3 &= b_4 & x_4 &= b_1 + b_2 + b_3 + b_4
\end{align*}
$$
9 The cyclic difference matrix $C$ has a line of solutions (in 4 dimensions) to $Cx = 0$:

\[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\text{ when } x = \begin{bmatrix}
c \\
c \\
c \\
c
\end{bmatrix} = \text{ any constant vector.}
\]

\[
z_2 - z_1 = b_1 \quad z_1 = -b_1 - b_2 - b_3 = \begin{bmatrix}
-1 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
= \Delta^{-1}b
\]

10 The forward differences of the squares are $(t + 1)^2 - t^2 = 2t + 1 - t^2 = 2t + 1$.

Differences of the $n$th power are $(t + 1)^n - t^n = t^n - nt^{n-1} + \cdots$. The leading term is the derivative $nt^{n-1}$. The binomial theorem gives all the terms of $(t + 1)^n$.

11 The centered difference matrices of even size seem to be invertible. Look at eqns. 1 and 4:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}
\text{ First solve }
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
-b_2 - b_4 \\
b_1 \\
-b_4 \\
b_1 + b_3
\end{bmatrix}
\]

12\text{ Odd size: The five centered difference equations lead to } b_1 + b_3 + b_5 = 0.

\[
x_2 = b_1 \quad \text{ Add equations 1, 3, 5}
\]

The left side of the sum is zero.

\[
x_3 - x_1 = b_2 \\
x_4 - x_2 = b_3 \\
x_5 - x_3 = b_4 \\
-x_4 = b_5
\]

There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. We are given that the ratios $a/c$ and $b/d$ are equal. Then $ad = bc$. Then (when you divide by $bd$) the ratios $a/b$ and $c/d$ must also be equal!