INTRODUCTION

TO

LINEAR

ALGEBRA

Fifth Edition

MANUAL FOR INSTRUCTORS

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Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

2 $\boldsymbol{v}+\boldsymbol{w}=(2,3)$ and $\boldsymbol{v}-\boldsymbol{w}=(6,-1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0,0)$.

3 This problem gives the diagonals $\boldsymbol{v}+\boldsymbol{w}$ and $\boldsymbol{v}-\boldsymbol{w}$ of the parallelogram and asks for the sides: The opposite of Problem 2. In this example $\boldsymbol{v}=(3,3)$ and $\boldsymbol{w}=(2,-2)$.

4 $3\boldsymbol{v}+\boldsymbol{w}=(7,5)$ and $c\boldsymbol{v}+d\boldsymbol{w}=(2c+d,c+2d)$.

5 $\boldsymbol{u}+\boldsymbol{v}=(-2,3,1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}=(0,0,0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w}=($ add first answers$)=(-2,3,1)$. The vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are in the same plane because a combination gives $(0,0,0)$. Stated another way: $\boldsymbol{u}=-\boldsymbol{v}-\boldsymbol{w}$ is in the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$.

6 The components of every $c\boldsymbol{v}+d\boldsymbol{w}$ add to zero because the components of $\boldsymbol{v}$ and of $\boldsymbol{w}$ add to zero. $c=3$ and $d=9$ give $(3,3,-6)$. There is no solution to $c\boldsymbol{v}+d\boldsymbol{w}=(3,3,6)$ because $3+3+6$ is not zero.

7 The nine combinations $c(2,1)+d(0,1)$ with $c=0,1,2$ and $d=(0,1,2)$ will lie on a lattice. If we took all whole numbers c and d, the lattice would lie over the whole plane.

8 The other diagonal is $\boldsymbol{v}-\boldsymbol{w}$ (or else $\boldsymbol{w}-\boldsymbol{v}$). Adding diagonals gives $2\boldsymbol{v}$ (or $2\boldsymbol{w}$).

9 The fourth corner can be $(4,4)$ or $(4,0)$ or $(-2,2)$. Three possible parallelograms!

10 $\boldsymbol{i}-\boldsymbol{j}=(1,1,0)$ is in the base (x-y plane). $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}=(1,1,1)$ is the opposite corner from $(0,0,0)$. Points in the cube have $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$.

11 Four more corners $(1,1,0),(1,0,1),(0,1,1),(1,1,1)$. The center point is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Centers of faces are $(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},\frac{1}{2},1)$ and $(0,\frac{1}{2},\frac{1}{2}),(1,\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},0,\frac{1}{2}),(\frac{1}{2},1,\frac{1}{2})$.

12 The combinations of $\boldsymbol{i}=(1,0,0)$ and $\boldsymbol{i}+\boldsymbol{j}=(1,1,0)$ fill the xy plane in xyz space.

13 Sum $=$ zero vector. Sum $=-2{:}00$ vector $=8{:}00$ vector. $2{:}00$ is 30° from horizontal $=(\cos\frac{\pi}{6},\sin\frac{\pi}{6})=(\sqrt{3}/2,1/2)$.

14 Moving the origin to 6:00 adds $\boldsymbol{j}=(0,1)$ to every vector. So the sum of twelve vectors changes from $\mathbf{0}$ to $12\boldsymbol{j}=(0,12)$.

15 The point $\dfrac{3}{4}\boldsymbol{v} + \dfrac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\dfrac{1}{4}\boldsymbol{v} + \dfrac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u} = \dfrac{1}{2}\boldsymbol{v} + \dfrac{1}{2}\boldsymbol{w}$. The vector $\boldsymbol{v} + \boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

16 All combinations with $c + d = 1$ are on the line that passes through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $\boldsymbol{V} = -\boldsymbol{v} + 2\boldsymbol{w}$ is on that line but it is beyond $\boldsymbol{w}$.

17 All vectors $c\boldsymbol{v} + c\boldsymbol{w}$ are on the line passing through $(0,0)$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v} + \boldsymbol{w}$ and back beyond $(0,0)$. With $c \geq 0$, half of this line is removed, leaving a *ray* that starts at $(0,0)$.

18 The combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $0 \leq c \leq 1$ and $0 \leq d \leq 1$ *fill the parallelogram* with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v} = (1,0)$ and $\boldsymbol{w} = (0,1)$ then $c\boldsymbol{v} + d\boldsymbol{w}$ fills the unit square. But when $\boldsymbol{v} = (a,0)$ and $\boldsymbol{w} = (b,0)$ these combinations only fill a segment of a line.

19 With $c \geq 0$ and $d \geq 0$ we get the infinite "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v} = (1,0)$ and $\boldsymbol{w} = (0,1)$, then the cone is the whole quadrant $x \geq 0$, $y \geq 0$. *Question*: What if $\boldsymbol{w} = -\boldsymbol{v}$? The cone opens to a half-space. But the combinations of $\boldsymbol{v} = (1,0)$ and $\boldsymbol{w} = (-1,0)$ only fill a line.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ lies between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c \geq 0$, $d \geq 0$, $e \geq 0$, and $c + d + e = \mathbf{1}$.

21 The sum is $(\boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{w} - \boldsymbol{v}) + (\boldsymbol{u} - \boldsymbol{w}) =$ **zero vector**. Those three sides of a triangle are in the same plane!

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

23 *All* vectors are combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ as drawn (not in the same plane). Start by seeing that $c\boldsymbol{u} + d\boldsymbol{v}$ fills a plane, then adding $e\boldsymbol{w}$ fills all of $\mathbf{R}^3$.

24 The combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$ fill one plane. The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill another plane. Those planes meet in a *line*: *only the vectors* $c\boldsymbol{v}$ are in both planes.

25 (a) For a line, choose $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w} =$ any nonzero vector (b) For a plane, choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions. A combination like $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ is in the same plane.

26 Two equations come from the two components: $c + 3d = 14$ and $2c + d = 8$. *The solution is* $c = 2$ *and* $d = 4$. Then $2(1, 2) + 4(3, 1) = (14, 8)$.

27 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

28 There are **6** unknown numbers $v_1, v_2, v_3, w_1, w_2, w_3$. The six equations come from the components of $\boldsymbol{v} + \boldsymbol{w} = (4, 5, 6)$ and $\boldsymbol{v} - \boldsymbol{w} = (2, 5, 8)$. Add to find $2\boldsymbol{v} = (6, 10, 14)$ so $\boldsymbol{v} = (3, 5, 7)$ and $\boldsymbol{w} = (1, 0, -1)$.

29 Two combinations out of infinitely many that produce $\boldsymbol{b} = (0, 1)$ are $-2\boldsymbol{u} + \boldsymbol{v}$ and $\frac{1}{2}\boldsymbol{w} - \frac{1}{2}\boldsymbol{v}$. **No**, three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the x-y plane could fail to produce $\boldsymbol{b}$ if all three lie on a line that does not contain $\boldsymbol{b}$. *Yes*, if one combination produces $\boldsymbol{b}$ then two (and infinitely many) combinations will produce $\boldsymbol{b}$. This is true even if $\boldsymbol{u} = \boldsymbol{0}$; the combinations can have different $c\boldsymbol{u}$.

30 The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill the plane *unless* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *lie on the same line through* $(0, 0)$. Four vectors whose combinations fill 4-dimensional space: one example is the "standard basis" $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$.

31 The equations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$ are

$$\begin{array}{rrrl} 2c & -d & & = 1 \\ -c & +2d & -e & = 0 \\ & -d & +2e & = 0 \end{array} \qquad \begin{array}{l} \text{So } d = 2e \\ \text{then } c = 3e \\ \text{then } 4e = 1 \end{array} \qquad \begin{array}{l} c = 3/4 \\ d = 2/4 \\ e = 1/4 \end{array}$$

Problem Set 1.2, page 18

1 $\boldsymbol{u} \cdot \boldsymbol{v} = -2.4 + 2.4 = 0$, $\boldsymbol{u} \cdot \boldsymbol{w} = -.6 + 1.6 = 1$, $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w} = 0 + 1$, $\boldsymbol{w} \cdot \boldsymbol{v} = 4 - 6 = -2 = \boldsymbol{v} \cdot \boldsymbol{w}$.

2 $\|\boldsymbol{u}\| = 1$ and $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = \sqrt{5}$. Then $|\boldsymbol{u} \cdot \boldsymbol{v}| = 0 < (1)(5)$ and $|\boldsymbol{v} \cdot \boldsymbol{w}| = 10 < 5\sqrt{5}$, confirming the Schwarz inequality.

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{4}{5}, \frac{3}{5}) = (0.8, 0.6)$. The vectors $\boldsymbol{w}, (2,-1)$, and $-\boldsymbol{w}$ make $0°, 90°, 180°$ angles with $\boldsymbol{w}$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (1/\sqrt{5}, 2/\sqrt{5})$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = 10/5\sqrt{5}$.

4 (a) $\boldsymbol{v} \cdot (-\boldsymbol{v}) = \mathbf{-1}$ (b) $(\boldsymbol{v}+\boldsymbol{w}) \cdot (\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot\boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = \mathbf{0}$ so $\theta = 90°$ (notice $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$) (c) $(\boldsymbol{v}-2\boldsymbol{w})\cdot(\boldsymbol{v}+2\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} - 4\boldsymbol{w}\cdot\boldsymbol{w} = 1 - 4 = \mathbf{-3}$.

5 $\boldsymbol{u}_1 = \boldsymbol{v}/\|\boldsymbol{v}\| = (1,3)/\sqrt{10}$ and $\boldsymbol{u}_2 = \boldsymbol{w}/\|\boldsymbol{w}\| = (2,1,2)/3$. $\boldsymbol{U}_1 = (3,-1)/\sqrt{10}$ is perpendicular to $\boldsymbol{u}_1$ (and so is $(-3,1)/\sqrt{10}$). $\boldsymbol{U}_2$ could be $(1,-2,0)/\sqrt{5}$: There is a whole plane of vectors perpendicular to $\boldsymbol{u}_2$, and a whole circle of unit vectors in that plane.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v}$. They lie on a line. All vectors (x,y,z) with $x+y+z=0$ lie on a *plane*. All vectors perpendicular to $(1,1,1)$ and $(1,2,3)$ lie on a *line* in 3-dimensional space.

7 (a) $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = 1/(2)(1)$ so $\theta = 60°$ or $\pi/3$ radians (b) $\cos\theta = 0$ so $\theta = 90°$ or $\pi/2$ radians (c) $\cos\theta = 2/(2)(2) = 1/2$ so $\theta = 60°$ or $\pi/3$ (d) $\cos\theta = -1/\sqrt{2}$ so $\theta = 135°$ or $3\pi/4$.

8 (a) False: $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors in the plane perpendicular to $\boldsymbol{u}$ (b) True: $\boldsymbol{u}\cdot(\boldsymbol{v}+2\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + 2\boldsymbol{u}\cdot\boldsymbol{w} = 0$ (c) True, $\|\boldsymbol{u}-\boldsymbol{v}\|^2 = (\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})$ splits into $\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} = \mathbf{2}$ when $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u} = 0$.

9 If $v_2w_2/v_1w_1 = -1$ then $v_2w_2 = -v_1w_1$ or $v_1w_1 + v_2w_2 = \boldsymbol{v}\cdot\boldsymbol{w} = 0$: perpendicular! The vectors $(1,4)$ and $(1,-\frac{1}{4})$ are perpendicular.

10 Slopes $2/1$ and $-1/2$ multiply to give -1: then $\boldsymbol{v}\cdot\boldsymbol{w} = 0$ and the vectors (the directions) are perpendicular.

11 $\boldsymbol{v}\cdot\boldsymbol{w} < 0$ means angle $> 90°$; these $\boldsymbol{w}$'s fill half of 3-dimensional space.

12 $(1,1)$ perpendicular to $(1,5) - c(1,1)$ if $(1,1)\cdot(1,5) - c(1,1)\cdot(1,1) = 6 - 2c = 0$ or $\boldsymbol{c = 3}$; $\boldsymbol{v}\cdot(\boldsymbol{w} - c\boldsymbol{v}) = 0$ if $c = \boldsymbol{v}\cdot\boldsymbol{w}/\boldsymbol{v}\cdot\boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to constructing a perpendicular vector.

13 The plane perpendicular to $(1,0,1)$ contains all vectors $(c,d,-c)$. In that plane, $\boldsymbol{v} = (1,0,-1)$ and $\boldsymbol{w} = (0,1,0)$ are perpendicular.

14 One possibility among many: $\boldsymbol{u} = (1,-1,0,0), \boldsymbol{v} = (0,0,1,-1), \boldsymbol{w} = (1,1,-1,-1)$ and $(1,1,1,1)$ are perpendicular to each other. "We can rotate those $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ in their 3D hyperplane and they will stay perpendicular."

15 $\frac{1}{2}(x+y) = (2+8)/2 = 5$ and $5 > 4$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = \mathbf{8/10}$.

16 $\|\boldsymbol{v}\|^2 = 1+1+\cdots+1 = 9$ so $\|\boldsymbol{v}\| = \mathbf{3}$; $\boldsymbol{u} = \boldsymbol{v}/3 = (\frac{1}{3},\ldots,\frac{1}{3})$ is a unit vector in 9D; $\boldsymbol{w} = (1,-1,0,\ldots,0)/\sqrt{2}$ is a unit vector in the 8D hyperplane perpendicular to $\boldsymbol{v}$.

17 $\cos\alpha = 1/\sqrt{2}$, $\cos\beta = 0$, $\cos\gamma = -1/\sqrt{2}$. For any vector $\boldsymbol{v} = (\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3)$ the cosines with $(1,0,0)$ and $(0,0,1)$ are $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = (v_1^2+v_2^2+v_3^2)/\|\boldsymbol{v}\|^2 = 1$.

18 $\|\boldsymbol{v}\|^2 = 4^2+2^2 = 20$ and $\|\boldsymbol{w}\|^2 = (-1)^2+2^2 = 5$. Pythagoras is $\|(3,4)\|^2 = 25 = 20+5$ for the length of the hypotenuse $\boldsymbol{v}+\boldsymbol{w} = (3,4)$.

19 Start from the rules $(1),(2),(3)$ for $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$ and $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w})$ and $(c\boldsymbol{v})\cdot\boldsymbol{w}$. Use rule (2) for $(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{v}+(\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{w}$. By rule (1) this is $\boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w})+\boldsymbol{w}\cdot(\boldsymbol{v}+\boldsymbol{w})$. Rule (2) again gives $\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w} = \boldsymbol{v}\cdot\boldsymbol{v}+2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w}$. Notice $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$! The main point is to feel free to open up parentheses.

20 We know that $(\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v}-2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w}$. The Law of Cosines writes $\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$ for $\boldsymbol{v}\cdot\boldsymbol{w}$. Here θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. When $\theta < 90°$ this $\boldsymbol{v}\cdot\boldsymbol{w}$ is positive, so in this case $\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w}$ is larger than $\|\boldsymbol{v}-\boldsymbol{w}\|^2$.

Pythagoras changes from equality $a^2+b^2 = c^2$ to *inequality* when $\theta < 90\,°$ or $\theta > 90\,°$.

21 $2\boldsymbol{v}\cdot\boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v}+\boldsymbol{w}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v}+2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|^2+2\|\boldsymbol{v}\|\|\boldsymbol{w}\|+\|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\|+\|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v}+\boldsymbol{w}\| \le \|\boldsymbol{v}\|+\|\boldsymbol{w}\|$.

22 $v_1^2w_1^2+2v_1w_1v_2w_2+v_2^2w_2^2 \le v_1^2w_1^2+v_1^2w_2^2+v_2^2w_1^2+v_2^2w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2w_2^2+v_2^2w_1^2-2v_1w_1v_2w_2$ which is $(v_1w_2-v_2w_1)^2 \ge 0$.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta-a) = \cos\beta\cos\alpha+\sin\beta\sin\alpha = v_1w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\|+v_2w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. This is $\cos\theta$ because $\beta-\alpha = \theta$.

24 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$. True: $.96 < 1$.

25 The cosine of θ is $x/\sqrt{x^2+y^2}$, near side over hypotenuse. Then $|\cos\theta|^2$ is not greater than 1: $x^2/(x^2+y^2) \le 1$.

26 The vectors $\boldsymbol{w} = (x, y)$ with $(1,2) \cdot \boldsymbol{w} = x + 2y = 5$ lie on a line in the xy plane. The shortest $\boldsymbol{w}$ on that line is $(1, 2)$. (The Schwarz inequality $\|\boldsymbol{w}\| \ge \boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\| = \sqrt{5}$ is an equality when $\cos\theta = 0$ and $\boldsymbol{w} = (1, 2)$ and $\|\boldsymbol{w}\| = \sqrt{5}$.)

27 The length $\|\boldsymbol{v} - \boldsymbol{w}\|$ is between 2 and 8 (triangle inequality when $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$). The dot product $\boldsymbol{v} \cdot \boldsymbol{w}$ is between -15 and 15 by the Schwarz inequality.

28 Three vectors in the plane could make angles greater than $90°$ with each other: for example $(1, 0), (-1, 4), (-1, -4)$. Four vectors could *not* do this ($360°$ total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$? Ben Harris and Greg Marks showed me that the answer is $n + 1$. The vectors from the center of a regular simplex in $\mathbf{R}^n$ to its $n + 1$ vertices all have negative dot products. If $n+2$ vectors in $\mathbf{R}^n$ had negative dot products, project them onto the plane orthogonal to the last one. Now you have $n + 1$ vectors in $\mathbf{R}^{n-1}$ with negative dot products. Keep going to 4 vectors in $\mathbf{R}^2$: no way!

29 For a specific example, pick $\boldsymbol{v} = (1, 2, -3)$ and then $\boldsymbol{w} = (-3, 1, 2)$. In this example $\cos\theta = \boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -7/\sqrt{14}\sqrt{14} = -1/2$ and $\theta = 120°$. This always happens when $x + y + z = 0$:

$$\boldsymbol{v} \cdot \boldsymbol{w} = xz + xy + yz = \frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2)$$

This is the same as $\boldsymbol{v} \cdot \boldsymbol{w} = 0 - \dfrac{1}{2}\,\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. Then $\cos\theta = \dfrac{1}{2}$.

30 Wikipedia gives this proof of geometric mean $G = \sqrt[3]{xyz} \le$ arithmetic mean $A = (x + y + z)/3$. First there is equality in case $x = y = z$. Otherwise A is somewhere between the three positive numbers, say for example $z < A < y$.

Use the known inequality $g \le a$ for the *two* positive numbers x and $y + z - A$. Their mean $a = \frac{1}{2}(x + y + z - A)$ is $\frac{1}{2}(3A - A)$ = same as A! So $a \ge g$ says that

$A^3 \geq g^2 A = x(y+z-A)A$. But $(y+z-A)A = (y-A)(A-z) + yz > yz$. Substitute to find $A^3 > xyz = G^3$ as we wanted to prove. Not easy!

There are many proofs of $G = (x_1 x_2 \cdots x_n)^{1/n} \leq A = (x_1 + x_2 + \cdots + x_n)/n$. In calculus you are maximizing G on the plane $x_1 + x_2 + \cdots + x_n = n$. The maximum occurs when all x's are equal.

31 The columns of the 4 by 4 "Hadamard matrix" (times $\frac{1}{2}$) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

32 The commands $V = \textbf{randn}\,(3,30); D = \textbf{sqrt}\,(\textbf{diag}\,(V' * V)); U = V \backslash D;$ will give 30 random unit vectors in the columns of U. Then $u' * U$ is a row matrix of 30 dot products whose average absolute value may be close to $2/\pi$.

Problem Set 1.3, page 29

1 $2\boldsymbol{s}_1 + 3\boldsymbol{s}_2 + 4\boldsymbol{s}_3 = (2,5,9)$. The same vector $\boldsymbol{b}$ comes from S times $\boldsymbol{x} = (2,3,4)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} (\text{row } 1) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}.$$

2 The solutions are $y_1 = 1$, $y_2 = 0$, $y_3 = 0$ (right side = column 1) and $y_1 = 1$, $y_2 = 3$, $y_3 = 5$. That second example illustrates that the first n odd numbers add to n^2.

3
$$\begin{aligned} y_1 &= B_1 \\ y_1 + y_2 &= B_2 \\ y_1 + y_2 + y_3 &= B_3 \end{aligned} \quad \text{gives} \quad \begin{aligned} y_1 &= B_1 \\ y_2 &= -B_1 + B_2 \\ y_3 &= -B_2 + B_3 \end{aligned} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

The inverse of $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$: **independent** columns in A and S!

4 The combination $0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 0\boldsymbol{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\boldsymbol{w}_2 = (\boldsymbol{w}_1 + \boldsymbol{w}_3)/2$ so one combination that gives zero is $\frac{1}{2}\boldsymbol{w}_1 - \boldsymbol{w}_2 + \frac{1}{2}\boldsymbol{w}_3 = \mathbf{0}$.

5 The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $\boldsymbol{r}_2 = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual. Two solutions to $y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + y_3\boldsymbol{r}_3 = \mathbf{0}$ are $(Y_1, Y_2, Y_3) = (1, -2, 1)$ and $(2, -4, 2)$.

6 $c = \mathbf{3}$ $\quad \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & \mathbf{3} \end{bmatrix}$ has column $3 =$ column $1 -$ column 2

$c = \mathbf{-1}$ $\quad \begin{bmatrix} 1 & 0 & \mathbf{-1} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has column $3 = -$ column $1 +$ column 2

$c = \mathbf{0}$ $\quad \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$ has column $3 = 3$ (column 1) $-$ column 2

7 All three rows are perpendicular to the solution $\boldsymbol{x}$ (the three equations $\boldsymbol{r}_1 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_2 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_3 \cdot \boldsymbol{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to $\boldsymbol{x}$ (the plane is also perpendicular to all multiples $c\boldsymbol{x}$).

8 $$\begin{aligned} x_1 - 0 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3 \\ x_4 - x_3 &= b_4 \end{aligned} \qquad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \\ x_4 &= b_1 + b_2 + b_3 + b_4 \end{aligned} \quad = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = A^{-1}\boldsymbol{b}$$

9 The cyclic difference matrix C has a line of solutions (in 4 dimensions) to $C\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{ any constant vector.}$$

10
$$\begin{aligned} z_2 - z_1 &= b_1 \\ z_3 - z_2 &= b_2 \\ 0 - z_3 &= b_3 \end{aligned} \qquad \begin{aligned} z_1 &= -b_1 - b_2 - b_3 \\ z_2 &= - b_2 - b_3 \\ z_3 &= - b_3 \end{aligned} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1}\boldsymbol{b}$$

11 The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the nth power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative nt^{n-1}. The binomial theorem gives all the terms of $(t+1)^n$.

12 Centered difference matrices of *even size* seem to be invertible. Look at eqns. 1 and 4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{matrix} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

13 *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ - x_4 &= b_5 \end{aligned}$$

Add equations $1, 3, 5$

The left side of the sum is zero

The right side is $b_1 + b_3 + b_5$

There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

14 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. We are given that the ratios a/c and b/d are equal. Then $ad = bc$. Then (when you divide by bd) the ratios a/b and c/d must also be equal!