

## Chapter 6

# Eigenvalues and Eigenvectors

### 6.1 Introduction to Eigenvalues

Linear equations  $A\mathbf{x} = \mathbf{b}$  come from steady state problems. Eigenvalues have their greatest importance in *dynamic problems*. The solution of  $d\mathbf{u}/dt = A\mathbf{u}$  is changing with time—growing or decaying or oscillating. We can't find it by elimination. This chapter enters a new part of linear algebra, based on  $A\mathbf{x} = \lambda\mathbf{x}$ . All matrices in this chapter are square.

A good model comes from the powers  $A, A^2, A^3, \dots$  of a matrix. Suppose you need the hundredth power  $A^{100}$ . The starting matrix  $A$  becomes unrecognizable after a few steps, and  $A^{100}$  is very close to  $\begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$ :

$$\begin{array}{ccccccc} \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} & \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} & \begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix} & \dots & \begin{bmatrix} .6000 & .6000 \\ .4000 & .4000 \end{bmatrix} \\ A & A^2 & A^3 & & A^{100} \end{array}$$

$A^{100}$  was found by using the *eigenvalues* of  $A$ , not by multiplying 100 matrices. Those eigenvalues (here they are 1 and  $1/2$ ) are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by  $A$ . ***Certain exceptional vectors  $\mathbf{x}$  are in the same direction as  $A\mathbf{x}$ . Those are the “eigenvectors”.*** Multiply an eigenvector by  $A$ , and the vector  $A\mathbf{x}$  is a number  $\lambda$  times the original  $\mathbf{x}$ .

**The basic equation is  $A\mathbf{x} = \lambda\mathbf{x}$ . The number  $\lambda$  is an eigenvalue of  $A$ .**

The eigenvalue  $\lambda$  tells whether the special vector  $\mathbf{x}$  is stretched or shrunk or reversed or left unchanged—when it is multiplied by  $A$ . We may find  $\lambda = 2$  or  $\frac{1}{2}$  or  $-1$  or  $1$ . The eigenvalue  $\lambda$  could be zero! Then  $A\mathbf{x} = 0\mathbf{x}$  means that this eigenvector  $\mathbf{x}$  is in the nullspace.

If  $A$  is the identity matrix, every vector has  $A\mathbf{x} = \mathbf{x}$ . All vectors are eigenvectors of  $I$ . All eigenvalues “lambda” are  $\lambda = 1$ . This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. We will show that  $\det(A - \lambda I) = 0$ .

This section will explain how to compute the  $\mathbf{x}$ 's and  $\lambda$ 's. It can come early in the course because we only need the determinant of a 2 by 2 matrix. Let me use  $\det(A - \lambda I) = 0$  to find the eigenvalues for this first example, and then derive it properly in equation (3).

**Example 1** The matrix  $A$  has two eigenvalues  $\lambda = 1$  and  $\lambda = 1/2$ . Look at  $\det(A - \lambda I)$ :

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \det \begin{bmatrix} .8 - \lambda & .3 \\ .2 & .7 - \lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1) \left( \lambda - \frac{1}{2} \right).$$

I factored the quadratic into  $\lambda - 1$  times  $\lambda - \frac{1}{2}$ , to see the two eigenvalues  $\lambda = 1$  and  $\lambda = \frac{1}{2}$ . For those numbers, the matrix  $A - \lambda I$  becomes *singular* (zero determinant). The eigenvectors  $\mathbf{x}_1$  and  $\mathbf{x}_2$  are in the nullspaces of  $A - I$  and  $A - \frac{1}{2}I$ .

$(A - I)\mathbf{x}_1 = 0$  is  $A\mathbf{x}_1 = \mathbf{x}_1$  and the first eigenvector is  $(.6, .4)$ .

$(A - \frac{1}{2}I)\mathbf{x}_2 = 0$  is  $A\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$  and the second eigenvector is  $(1, -1)$ :

$$\mathbf{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad A\mathbf{x}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \mathbf{x}_1 \quad (A\mathbf{x} = \mathbf{x} \text{ means that } \lambda_1 = 1)$$

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad A\mathbf{x}_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \frac{1}{2}\mathbf{x}_2 \text{ so } \lambda_2 = \frac{1}{2}).$$

If  $\mathbf{x}_1$  is multiplied again by  $A$ , we still get  $\mathbf{x}_1$ . Every power of  $A$  will give  $A^n\mathbf{x}_1 = \mathbf{x}_1$ . Multiplying  $\mathbf{x}_2$  by  $A$  gave  $\frac{1}{2}\mathbf{x}_2$ , and if we multiply again we get  $(\frac{1}{2})^2$  times  $\mathbf{x}_2$ .

*When  $A$  is squared, the eigenvectors stay the same. The eigenvalues are squared.*

This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of  $A^{100}$  are the same  $\mathbf{x}_1$  and  $\mathbf{x}_2$ . The eigenvalues of  $A^{100}$  are  $1^{100} = 1$  and  $(\frac{1}{2})^{100} = \text{very small number}$ .

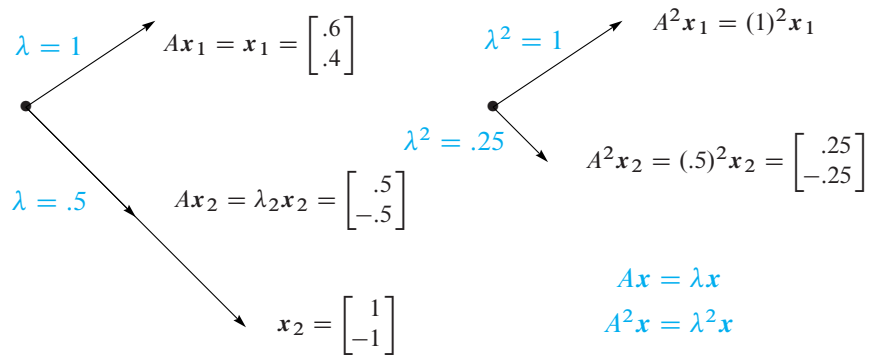


Figure 6.1: The eigenvectors keep their directions.  $A^2$  has eigenvalues  $1^2$  and  $(.5)^2$ .

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of  $A$  is the combination  $\mathbf{x}_1 + (.2)\mathbf{x}_2$ :

**Separate into eigenvectors** 
$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \mathbf{x}_1 + (.2)\mathbf{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}. \quad (1)$$

Multiplying by  $A$  gives  $(.7, .3)$ , the first column of  $A^2$ . Do it separately for  $\mathbf{x}_1$  and  $(.2)\mathbf{x}_2$ . Of course  $A\mathbf{x}_1 = \mathbf{x}_1$ . And  $A$  multiplies  $\mathbf{x}_2$  by its eigenvalue  $\frac{1}{2}$ :

$$\text{Multiply each } \mathbf{x}_i \text{ by } \lambda_i \quad A \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix} \quad \text{is} \quad \mathbf{x}_1 + \frac{1}{2}(.2)\mathbf{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .1 \\ -.1 \end{bmatrix}.$$

*Each eigenvector is multiplied by its eigenvalue*, when we multiply by  $A$ . We didn't need these eigenvectors to find  $A^2$ . But it is the good way to do 99 multiplications. At every step  $\mathbf{x}_1$  is unchanged and  $\mathbf{x}_2$  is multiplied by  $(\frac{1}{2})$ , so we have  $(\frac{1}{2})^{99}$ :

$$A^{99} \begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is really} \quad \mathbf{x}_1 + (.2)\left(\frac{1}{2}\right)^{99}\mathbf{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}.$$

This is the first column of  $A^{100}$ . The number we originally wrote as .6000 was not exact. We left out  $(.2)(\frac{1}{2})^{99}$  which wouldn't show up for 30 decimal places.

The eigenvector  $\mathbf{x}_1$  is a "steady state" that doesn't change (because  $\lambda_1 = 1$ ). The eigenvector  $\mathbf{x}_2$  is a "decaying mode" that virtually disappears (because  $\lambda_2 = .5$ ). The higher the power of  $A$ , the closer its columns approach the steady state.

We mention that this particular  $A$  is a **Markov matrix**. Its entries are positive and every column adds to 1. Those facts guarantee that the largest eigenvalue is  $\lambda = 1$  (as we found). Its eigenvector  $\mathbf{x}_1 = (.6, .4)$  is the *steady state*—which all columns of  $A^k$  will approach. Section 8.3 shows how Markov matrices appear in applications like Google.

For projections we can spot the steady state ( $\lambda = 1$ ) and the nullspace ( $\lambda = 0$ ).

**Example 2** The projection matrix  $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$  has eigenvalues  $\lambda = 1$  and  $\lambda = 0$ .

Its eigenvectors are  $\mathbf{x}_1 = (1, 1)$  and  $\mathbf{x}_2 = (1, -1)$ . For those vectors,  $P\mathbf{x}_1 = \mathbf{x}_1$  (steady state) and  $P\mathbf{x}_2 = \mathbf{0}$  (nullspace). This example illustrates Markov matrices and singular matrices and (most important) symmetric matrices. All have special  $\lambda$ 's and  $\mathbf{x}$ 's:

1. Each column of  $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$  adds to 1, so  $\lambda = 1$  is an eigenvalue.
2.  $P$  is **singular**, so  $\lambda = 0$  is an eigenvalue.
3.  $P$  is **symmetric**, so its eigenvectors  $(1, 1)$  and  $(1, -1)$  are perpendicular.

The only eigenvalues of a projection matrix are 0 and 1. The eigenvectors for  $\lambda = 0$  (which means  $P\mathbf{x} = 0\mathbf{x}$ ) fill up the nullspace. The eigenvectors for  $\lambda = 1$  (which means  $P\mathbf{x} = \mathbf{x}$ ) fill up the column space. The nullspace is projected to zero. The column space projects onto itself. The projection keeps the column space and destroys the nullspace:

$$\text{Project each part} \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{projects onto} \quad P\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

*Special properties of a matrix lead to special eigenvalues and eigenvectors.* That is a major theme of this chapter (it is captured in a table at the very end).

Projections have  $\lambda = 0$  and  $1$ . Permutations have all  $|\lambda| = 1$ . The next matrix  $R$  (a reflection and at the same time a permutation) is also special.

**Example 3** The reflection matrix  $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  has eigenvalues  $1$  and  $-1$ .

The eigenvector  $(1, 1)$  is unchanged by  $R$ . The second eigenvector is  $(1, -1)$ —its signs are reversed by  $R$ . A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for  $R$  are the same as for  $P$ , because *reflection* =  $2(\text{projection}) - I$ :

$$R = 2P - I \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2 \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (2)$$

Here is the point. If  $Px = \lambda x$  then  $2Px = 2\lambda x$ . The eigenvalues are doubled when the matrix is doubled. Now subtract  $Ix = x$ . The result is  $(2P - I)x = (2\lambda - 1)x$ . **When a matrix is shifted by  $I$ , each  $\lambda$  is shifted by  $1$ .** No change in eigenvectors.

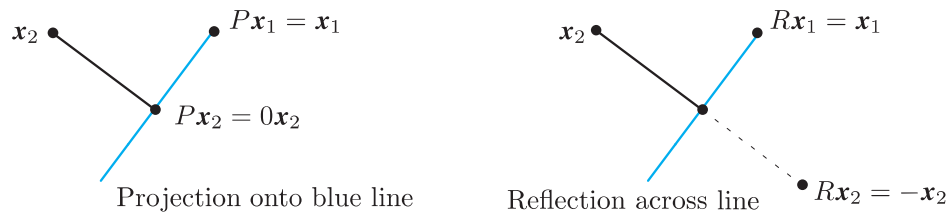


Figure 6.2: Projections  $P$  have eigenvalues  $1$  and  $0$ . Reflections  $R$  have  $\lambda = 1$  and  $-1$ . A typical  $x$  changes direction, but not the eigenvectors  $x_1$  and  $x_2$ .

Key idea: The eigenvalues of  $R$  and  $P$  are related exactly as the matrices are related:

The eigenvalues of  $R = 2P - I$  are  $2(1) - 1 = 1$  and  $2(0) - 1 = -1$ .

The eigenvalues of  $R^2$  are  $\lambda^2$ . In this case  $R^2 = I$ . Check  $(1)^2 = 1$  and  $(-1)^2 = 1$ .

### The Equation for the Eigenvalues

For projections and reflections we found  $\lambda$ 's and  $x$ 's by geometry:  $Px = x$ ,  $Px = \mathbf{0}$ ,  $Rx = -x$ . Now we use determinants and linear algebra. *This is the key calculation in the chapter*—almost every application starts by solving  $Ax = \lambda x$ .

First move  $\lambda x$  to the left side. Write the equation  $Ax = \lambda x$  as  $(A - \lambda I)x = \mathbf{0}$ . The matrix  $A - \lambda I$  times the eigenvector  $x$  is the zero vector. **The eigenvectors make up the nullspace of  $A - \lambda I$ .** When we know an eigenvalue  $\lambda$ , we find an eigenvector by solving  $(A - \lambda I)x = \mathbf{0}$ .

Eigenvalues first. If  $(A - \lambda I)x = \mathbf{0}$  has a nonzero solution,  $A - \lambda I$  is not invertible. **The determinant of  $A - \lambda I$  must be zero.** This is how to recognize an eigenvalue  $\lambda$ :

**Eigenvalues** The number  $\lambda$  is an eigenvalue of  $A$  if and only if  $A - \lambda I$  is singular:

$$\det(A - \lambda I) = 0. \quad (3)$$

This “characteristic equation”  $\det(A - \lambda I) = 0$  involves only  $\lambda$ , not  $\mathbf{x}$ . When  $A$  is  $n$  by  $n$ , the equation has degree  $n$ . Then  $A$  has  $n$  eigenvalues and each  $\lambda$  leads to  $\mathbf{x}$ :

**For each  $\lambda$  solve  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  or  $A\mathbf{x} = \lambda\mathbf{x}$  to find an eigenvector  $\mathbf{x}$ .**

**Example 4**  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$  is already singular (zero determinant). Find its  $\lambda$ 's and  $\mathbf{x}$ 's.

When  $A$  is singular,  $\lambda = 0$  is one of the eigenvalues. The equation  $A\mathbf{x} = 0\mathbf{x}$  has solutions. They are the eigenvectors for  $\lambda = 0$ . But  $\det(A - \lambda I) = 0$  is the way to find *all*  $\lambda$ 's and  $\mathbf{x}$ 's. Always subtract  $\lambda I$  from  $A$ :

$$\text{Subtract } \lambda \text{ from the diagonal to find } A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix}. \quad (4)$$

**Take the determinant “ $ad - bc$ ” of this 2 by 2 matrix.** From  $1 - \lambda$  times  $4 - \lambda$ , the “ $ad$ ” part is  $\lambda^2 - 5\lambda + 4$ . The “ $bc$ ” part, not containing  $\lambda$ , is 2 times 2.

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - (2)(2) = \lambda^2 - 5\lambda. \quad (5)$$

**Set this determinant  $\lambda^2 - 5\lambda$  to zero.** One solution is  $\lambda = 0$  (as expected, since  $A$  is singular). Factoring into  $\lambda$  times  $\lambda - 5$ , the other root is  $\lambda = 5$ :

$$\det(A - \lambda I) = \lambda^2 - 5\lambda = 0 \quad \text{yields the eigenvalues } \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 5.$$

Now find the eigenvectors. Solve  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  separately for  $\lambda_1 = 0$  and  $\lambda_2 = 5$ :

$$(A - 0I)\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ for } \lambda_1 = 0$$

$$(A - 5I)\mathbf{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } \lambda_2 = 5.$$

The matrices  $A - 0I$  and  $A - 5I$  are singular (because 0 and 5 are eigenvalues). The eigenvectors  $(2, -1)$  and  $(1, 2)$  are in the nullspaces:  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  is  $A\mathbf{x} = \lambda\mathbf{x}$ .

We need to emphasize: *There is nothing exceptional about  $\lambda = 0$ .* Like every other number, zero might be an eigenvalue and it might not. If  $A$  is singular, it is. The eigenvectors fill the nullspace:  $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$ . If  $A$  is invertible, zero is not an eigenvalue. We shift  $A$  by a multiple of  $I$  to *make it singular*.

In the example, the shifted matrix  $A - 5I$  is singular and 5 is the other eigenvalue.

**Summary** To solve the eigenvalue problem for an  $n$  by  $n$  matrix, follow these steps:

1. **Compute the determinant of  $A - \lambda I$ .** With  $\lambda$  subtracted along the diagonal, this determinant starts with  $\lambda^n$  or  $-\lambda^n$ . It is a polynomial in  $\lambda$  of degree  $n$ .
2. **Find the roots of this polynomial,** by solving  $\det(A - \lambda I) = 0$ . The  $n$  roots are the  $n$  eigenvalues of  $A$ . They make  $A - \lambda I$  singular.
3. For each eigenvalue  $\lambda$ , **solve  $(A - \lambda I)x = 0$  to find an eigenvector  $x$ .**

A note on the eigenvectors of 2 by 2 matrices. When  $A - \lambda I$  is singular, both rows are multiples of a vector  $(a, b)$ . *The eigenvector is any multiple of  $(b, -a)$ .* The example had  $\lambda = 0$  and  $\lambda = 5$ :

$\lambda = 0$  : rows of  $A - 0I$  in the direction  $(1, 2)$ ; eigenvector in the direction  $(2, -1)$

$\lambda = 5$  : rows of  $A - 5I$  in the direction  $(-4, 2)$ ; eigenvector in the direction  $(2, 4)$ .

Previously we wrote that last eigenvector as  $(1, 2)$ . Both  $(1, 2)$  and  $(2, 4)$  are correct. There is a whole *line of eigenvectors*—any nonzero multiple of  $x$  is as good as  $x$ . MATLAB's `eig(A)` divides by the length, to make the eigenvector into a unit vector.

We end with a warning. Some 2 by 2 matrices have only *one* line of eigenvectors. This can only happen when two eigenvalues are equal. (On the other hand  $A = I$  has equal eigenvalues and plenty of eigenvectors.) Similarly some  $n$  by  $n$  matrices don't have  $n$  independent eigenvectors. Without  $n$  eigenvectors, we don't have a basis. We can't write every  $v$  as a combination of eigenvectors. In the language of the next section, we can't diagonalize a matrix without  $n$  independent eigenvectors.

### Good News, Bad News

Bad news first: If you add a row of  $A$  to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the  $\lambda$ 's.* The triangular  $U$  has *its* eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of  $A$ ! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 7.$$

Good news second: The *product  $\lambda_1$  times  $\lambda_2$  and the sum  $\lambda_1 + \lambda_2$  can be found quickly from the matrix.* For this  $A$ , the product is 0 times 7. That agrees with the determinant (which is 0). The sum of eigenvalues is 0 + 7. That agrees with the sum down the main diagonal (the **trace** is 1 + 6). These quick checks always work:

*The product of the  $n$  eigenvalues equals the determinant.  
The sum of the  $n$  eigenvalues equals the sum of the  $n$  diagonal entries.*

The sum of the entries on the main diagonal is called the **trace** of  $A$ :

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace} = a_{11} + a_{22} + \cdots + a_{nn}. \quad (6)$$

Those checks are very useful. They are proved in Problems 16–17 and again in the next section. They don't remove the pain of computing  $\lambda$ 's. But when the computation is wrong, they generally tell us so. To compute the correct  $\lambda$ 's, go back to  $\det(A - \lambda I) = 0$ .

The determinant test makes the *product* of the  $\lambda$ 's equal to the *product* of the pivots (assuming no row exchanges). But the sum of the  $\lambda$ 's is not the sum of the pivots—as the example showed. The individual  $\lambda$ 's have almost nothing to do with the pivots. In this new part of linear algebra, the key equation is really *nonlinear*:  $\lambda$  multiplies  $\mathbf{x}$ .

### Why do the eigenvalues of a triangular matrix lie on its diagonal?

## Imaginary Eigenvalues

One more bit of news (not too terrible). The eigenvalues might not be real numbers.

**Example 5** The  $90^\circ$  rotation  $Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  has no real eigenvectors. Its eigenvalues are  $\lambda = i$  and  $\lambda = -i$ . Sum of  $\lambda$ 's = trace = 0. Product = determinant = 1.

After a rotation, *no vector*  $Q\mathbf{x}$  stays in the same direction as  $\mathbf{x}$  (except  $\mathbf{x} = \mathbf{0}$  which is useless). There cannot be an eigenvector, unless we go to *imaginary numbers*. Which we do.

To see how  $i$  can help, look at  $Q^2$  which is  $-I$ . If  $Q$  is rotation through  $90^\circ$ , then  $Q^2$  is rotation through  $180^\circ$ . Its eigenvalues are  $-1$  and  $-1$ . (Certainly  $-I\mathbf{x} = -1\mathbf{x}$ .) Squaring  $Q$  will square each  $\lambda$ , so we must have  $\lambda^2 = -1$ . The eigenvalues of the  $90^\circ$  rotation matrix  $Q$  are  $+i$  and  $-i$ , because  $i^2 = -1$ .

Those  $\lambda$ 's come as usual from  $\det(Q - \lambda I) = 0$ . This equation gives  $\lambda^2 + 1 = 0$ . Its roots are  $i$  and  $-i$ . We meet the imaginary number  $i$  also in the eigenvectors:

$$\text{Complex eigenvectors} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = i \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = -i \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Somehow these complex vectors  $\mathbf{x}_1 = (1, i)$  and  $\mathbf{x}_2 = (i, 1)$  keep their direction as they are rotated. Don't ask me how. This example makes the all-important point that real matrices can easily have complex eigenvalues and eigenvectors. The particular eigenvalues  $i$  and  $-i$  also illustrate two special properties of  $Q$ :

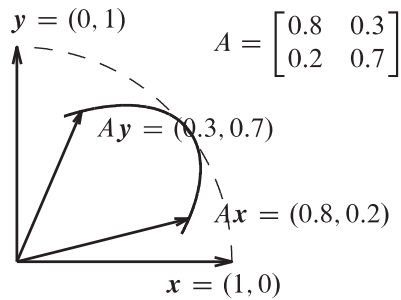
1.  $Q$  is an orthogonal matrix so the absolute value of each  $\lambda$  is  $|\lambda| = 1$ .
2.  $Q$  is a skew-symmetric matrix so each  $\lambda$  is pure imaginary.

A symmetric matrix ( $A^T = A$ ) can be compared to a real number. A skew-symmetric matrix ( $A^T = -A$ ) can be compared to an imaginary number. An orthogonal matrix ( $A^T A = I$ ) can be compared to a complex number with  $|\lambda| = 1$ . For the eigenvalues those are more than analogies—they are theorems to be proved in Section 6.4.

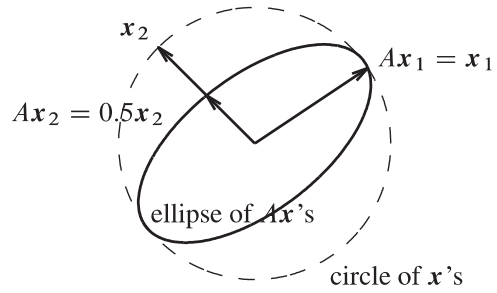
The eigenvectors for all these special matrices are perpendicular. Somehow  $(i, 1)$  and  $(1, i)$  are perpendicular (Chapter 10 explains the dot product of complex vectors).

### Eigshow in MATLAB

There is a MATLAB demo (just type **eigshow**), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector  $\mathbf{x} = (1, 0)$ . *The mouse makes this vector move around the unit circle.* At the same time the screen shows  $A\mathbf{x}$ , in color and also moving. Possibly  $A\mathbf{x}$  is ahead of  $\mathbf{x}$ . Possibly  $A\mathbf{x}$  is behind  $\mathbf{x}$ . *Sometimes  $A\mathbf{x}$  is parallel to  $\mathbf{x}$ .* At that parallel moment,  $A\mathbf{x} = \lambda\mathbf{x}$  (at  $\mathbf{x}_1$  and  $\mathbf{x}_2$  in the second figure).



These are not eigenvectors



$A\mathbf{x}$  lines up with  $\mathbf{x}$  at eigenvectors

The eigenvalue  $\lambda$  is the length of  $A\mathbf{x}$ , when the unit eigenvector  $\mathbf{x}$  lines up. The built-in choices for  $A$  illustrate three possibilities: 0, 1, or 2 directions where  $A\mathbf{x}$  crosses  $\mathbf{x}$ .

- 0. There are *no real eigenvectors*.  $A\mathbf{x}$  stays behind or ahead of  $\mathbf{x}$ . This means the eigenvalues and eigenvectors are complex, as they are for the rotation  $Q$ .
- 1. There is only *one* line of eigenvectors (unusual). The moving directions  $A\mathbf{x}$  and  $\mathbf{x}$  touch but don't cross over. This happens for the last 2 by 2 matrix below.
- 2. There are eigenvectors in *two* independent directions. This is typical!  $A\mathbf{x}$  crosses  $\mathbf{x}$  at the first eigenvector  $\mathbf{x}_1$ , and it crosses back at the second eigenvector  $\mathbf{x}_2$ . Then  $A\mathbf{x}$  and  $\mathbf{x}$  cross again at  $-\mathbf{x}_1$  and  $-\mathbf{x}_2$ .

You can mentally follow  $\mathbf{x}$  and  $A\mathbf{x}$  for these five matrices. Under the matrices I will count their real eigenvectors. Can you see where  $A\mathbf{x}$  lines up with  $\mathbf{x}$ ?

$$A = \begin{matrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ \mathbf{2} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{matrix}$$



When  $A$  is singular (rank one), its column space is a line. The vector  $A\mathbf{x}$  goes up and down that line while  $\mathbf{x}$  circles around. One eigenvector  $\mathbf{x}$  is along the line. Another eigenvector appears when  $A\mathbf{x}_2 = \mathbf{0}$ . Zero is an eigenvalue of a singular matrix.

### ■ REVIEW OF THE KEY IDEAS ■

1.  $A\mathbf{x} = \lambda\mathbf{x}$  says that eigenvectors  $\mathbf{x}$  keep the same direction when multiplied by  $A$ .
2.  $A\mathbf{x} = \lambda\mathbf{x}$  also says that  $\det(A - \lambda I) = 0$ . This determines  $n$  eigenvalues.
3. The eigenvalues of  $A^2$  and  $A^{-1}$  are  $\lambda^2$  and  $\lambda^{-1}$ , with the same eigenvectors.
4. The sum of the  $\lambda$ 's equals the sum down the main diagonal of  $A$  (*the trace*). The product of the  $\lambda$ 's equals the determinant.
5. Projections  $P$ , reflections  $R$ ,  $90^\circ$  rotations  $Q$  have special eigenvalues  $1, 0, -1, i, -i$ . Singular matrices have  $\lambda = 0$ . Triangular matrices have  $\lambda$ 's on their diagonal.

### ■ WORKED EXAMPLES ■

**6.1 A** Find the eigenvalues and eigenvectors of  $A$  and  $A^2$  and  $A^{-1}$  and  $A + 4I$ :

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace  $\lambda_1 + \lambda_2$  and the determinant  $\lambda_1\lambda_2$  for  $A$  and also  $A^2$ .

**Solution** The eigenvalues of  $A$  come from  $\det(A - \lambda I) = 0$ :

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into  $(\lambda - 1)(\lambda - 3) = 0$  so the eigenvalues of  $A$  are  $\lambda_1 = 1$  and  $\lambda_2 = 3$ . For the trace, the sum  $2 + 2$  agrees with  $1 + 3$ . The determinant  $3$  agrees with the product  $\lambda_1\lambda_2 = 3$ . The eigenvectors come separately by solving  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  which is  $A\mathbf{x} = \lambda\mathbf{x}$ :

$$\lambda = 1: \quad (A - I)\mathbf{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives the eigenvector } \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 3: \quad (A - 3I)\mathbf{x} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives the eigenvector } \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$A^2$  and  $A^{-1}$  and  $A + 4I$  keep the *same eigenvectors* as  $A$ . Their eigenvalues are  $\lambda^2$  and  $\lambda^{-1}$  and  $\lambda + 4$ :

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \quad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \quad A + 4I \text{ has } \frac{1+4=5}{3+4=7}$$

The trace of  $A^2$  is  $5 + 5$  which agrees with  $1 + 9$ . The determinant is  $25 - 16 = 9$ .

Notes for later sections:  $A$  has *orthogonal eigenvectors* (Section 6.4 on symmetric matrices).  $A$  can be *diagonalized* since  $\lambda_1 \neq \lambda_2$  (Section 6.2).  $A$  is *similar* to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.6).  $A$  is a *positive definite matrix* (Section 6.5) since  $A = A^T$  and the  $\lambda$ 's are positive.

**6.1 B** Find the eigenvalues and eigenvectors of this 3 by 3 matrix  $A$ :

**Symmetric matrix**

**Singular matrix**

**Trace  $1 + 2 + 1 = 4$**

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

**Solution** Since all rows of  $A$  add to zero, the vector  $\mathbf{x} = (1, 1, 1)$  gives  $A\mathbf{x} = \mathbf{0}$ . This is an eigenvector for the eigenvalue  $\lambda = 0$ . To find  $\lambda_2$  and  $\lambda_3$  I will compute the 3 by 3 determinant:

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda)(1 - \lambda) - 2(1 - \lambda) \\ = (1 - \lambda)[(2 - \lambda)(1 - \lambda) - 2] \\ = (1 - \lambda)(-\lambda)(3 - \lambda).$$

That factor  $-\lambda$  confirms that  $\lambda = 0$  is a root, and an eigenvalue of  $A$ . The other factors  $(1 - \lambda)$  and  $(3 - \lambda)$  give the other eigenvalues 1 and 3, adding to 4 (the trace). Each eigenvalue 0, 1, 3 corresponds to an eigenvector:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad A\mathbf{x}_1 = \mathbf{0}\mathbf{x}_1 \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad A\mathbf{x}_2 = \mathbf{1}\mathbf{x}_2 \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad A\mathbf{x}_3 = \mathbf{3}\mathbf{x}_3.$$

I notice again that eigenvectors are perpendicular when  $A$  is symmetric.

The 3 by 3 matrix produced a third-degree (cubic) polynomial for  $\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 3\lambda$ . We were lucky to find simple roots  $\lambda = 0, 1, 3$ . Normally we would use a command like **eig**( $A$ ), and the computation will never even use determinants (Section 9.3 shows a better way for large matrices).

The full command  $[S, D] = \mathbf{eig}(A)$  will produce unit eigenvectors in the columns of the **eigenvector matrix**  $S$ . The first one happens to have three minus signs, reversed from  $(1, 1, 1)$  and divided by  $\sqrt{3}$ . The eigenvalues of  $A$  will be on the diagonal of the **eigenvalue matrix** (typed as  $D$  but soon called  $\Lambda$ ).

### Problem Set 6.1

- 1 The example at the start of the chapter has powers of this matrix  $A$ :

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors.

- Show from  $A$  how a row exchange can produce different eigenvalues.
- Why is a zero eigenvalue *not* changed by the steps of elimination?

- 2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$  has the \_\_\_\_\_ eigenvectors as  $A$ . Its eigenvalues are \_\_\_\_\_ by 1.

- 3 Compute the eigenvalues and eigenvectors of  $A$  and  $A^{-1}$ . Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

$A^{-1}$  has the \_\_\_\_\_ eigenvectors as  $A$ . When  $A$  has eigenvalues  $\lambda_1$  and  $\lambda_2$ , its inverse has eigenvalues \_\_\_\_\_.

- 4 Compute the eigenvalues and eigenvectors of  $A$  and  $A^2$ :

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

$A^2$  has the same \_\_\_\_\_ as  $A$ . When  $A$  has eigenvalues  $\lambda_1$  and  $\lambda_2$ ,  $A^2$  has eigenvalues \_\_\_\_\_. In this example, why is  $\lambda_1^2 + \lambda_2^2 = 13$ ?

- 5 Find the eigenvalues of  $A$  and  $B$  (easy for triangular matrices) and  $A + B$ :

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of  $A + B$  (are equal to)(are not equal to) eigenvalues of  $A$  plus eigenvalues of  $B$ .

- 6 Find the eigenvalues of  $A$  and  $B$  and  $AB$  and  $BA$ :

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

- Are the eigenvalues of  $AB$  equal to eigenvalues of  $A$  times eigenvalues of  $B$ ?
- Are the eigenvalues of  $AB$  equal to the eigenvalues of  $BA$ ?

- 7 Elimination produces  $A = LU$ . The eigenvalues of  $U$  are on its diagonal; they are the \_\_\_\_\_. The eigenvalues of  $L$  are on its diagonal; they are all \_\_\_\_\_. The eigenvalues of  $A$  are not the same as \_\_\_\_\_.
- 8 (a) If you know that  $\mathbf{x}$  is an eigenvector, the way to find  $\lambda$  is to \_\_\_\_\_.  
 (b) If you know that  $\lambda$  is an eigenvalue, the way to find  $\mathbf{x}$  is to \_\_\_\_\_.
- 9 What do you do to the equation  $A\mathbf{x} = \lambda\mathbf{x}$ , in order to prove (a), (b), and (c)?
- (a)  $\lambda^2$  is an eigenvalue of  $A^2$ , as in Problem 4.  
 (b)  $\lambda^{-1}$  is an eigenvalue of  $A^{-1}$ , as in Problem 3.  
 (c)  $\lambda + 1$  is an eigenvalue of  $A + I$ , as in Problem 2.
- 10 Find the eigenvalues and eigenvectors for both of these Markov matrices  $A$  and  $A^\infty$ . Explain from those answers why  $A^{100}$  is close to  $A^\infty$ :

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

- 11 Here is a strange fact about 2 by 2 matrices with eigenvalues  $\lambda_1 \neq \lambda_2$ : The columns of  $A - \lambda_1 I$  are multiples of the eigenvector  $\mathbf{x}_2$ . Any idea why this should be?
- 12 Find three eigenvectors for this matrix  $P$  (projection matrices have  $\lambda = 1$  and 0):

$$\text{Projection matrix} \quad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same  $\lambda$ , so do all their linear combinations. Find an eigenvector of  $P$  with no zero components.

- 13 From the unit vector  $\mathbf{u} = \left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$  construct the rank one projection matrix  $P = \mathbf{u}\mathbf{u}^T$ . This matrix has  $P^2 = P$  because  $\mathbf{u}^T\mathbf{u} = 1$ .
- (a)  $P\mathbf{u} = \mathbf{u}$  comes from  $(\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u}(\text{_____})$ . Then  $\mathbf{u}$  is an eigenvector with  $\lambda = 1$ .  
 (b) If  $\mathbf{v}$  is perpendicular to  $\mathbf{u}$  show that  $P\mathbf{v} = \mathbf{0}$ . Then  $\lambda = 0$ .  
 (c) Find three independent eigenvectors of  $P$  all with eigenvalue  $\lambda = 0$ .
- 14 Solve  $\det(Q - \lambda I) = 0$  by the quadratic formula to reach  $\lambda = \cos \theta \pm i \sin \theta$ :

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{rotates the } xy \text{ plane by the angle } \theta. \text{ No real } \lambda\text{'s.}$$

Find the eigenvectors of  $Q$  by solving  $(Q - \lambda I)\mathbf{x} = \mathbf{0}$ . Use  $i^2 = -1$ .

- 15 Every permutation matrix leaves  $\mathbf{x} = (1, 1, \dots, 1)$  unchanged. Then  $\lambda = 1$ . Find two more  $\lambda$ 's (possibly complex) for these permutations, from  $\det(P - \lambda I) = 0$ :

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

- 16 **The determinant of  $A$  equals the product  $\lambda_1 \lambda_2 \cdots \lambda_n$ .** Start with the polynomial  $\det(A - \lambda I)$  separated into its  $n$  factors (always possible). Then set  $\lambda = 0$ :

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \quad \text{so} \quad \det A = \underline{\hspace{2cm}}.$$

Check this rule in Example 1 where the Markov matrix has  $\lambda = 1$  and  $\frac{1}{2}$ .

- 17 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues  $\lambda = (a + d + \sqrt{\quad})/2$  and  $\lambda = \underline{\hspace{2cm}}$ . Their sum is  $\underline{\hspace{2cm}}$ . If  $A$  has  $\lambda_1 = 3$  and  $\lambda_2 = 4$  then  $\det(A - \lambda I) = \underline{\hspace{2cm}}$ .

- 18 If  $A$  has  $\lambda_1 = 4$  and  $\lambda_2 = 5$  then  $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$ . Find three matrices that have trace  $a + d = 9$  and determinant 20 and  $\lambda = 4, 5$ .

- 19 A 3 by 3 matrix  $B$  is known to have eigenvalues 0, 1, 2. This information is enough to find three of these (give the answers where possible):

- the rank of  $B$
- the determinant of  $B^T B$
- the eigenvalues of  $B^T B$
- the eigenvalues of  $(B^2 + I)^{-1}$ .

- 20 Choose the last rows of  $A$  and  $C$  to give eigenvalues 4, 7 and 1, 2, 3:

**Companion matrices** 
$$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$

- 21 **The eigenvalues of  $A$  equal the eigenvalues of  $A^T$ .** This is because  $\det(A - \lambda I)$  equals  $\det(A^T - \lambda I)$ . That is true because  $\underline{\hspace{2cm}}$ . Show by an example that the eigenvectors of  $A$  and  $A^T$  are *not* the same.

- 22 Construct any 3 by 3 Markov matrix  $M$ : positive entries down each column add to 1. Show that  $M^T(1, 1, 1) = (1, 1, 1)$ . By Problem 21,  $\lambda = 1$  is also an eigenvalue of  $M$ . Challenge: A 3 by 3 singular Markov matrix with trace  $\frac{1}{2}$  has what  $\lambda$ 's?

- 23 Find three 2 by 2 matrices that have  $\lambda_1 = \lambda_2 = 0$ . The trace is zero and the determinant is zero.  $A$  might not be the zero matrix but check that  $A^2 = 0$ .
- 24 This matrix is singular with rank one. Find three  $\lambda$ 's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [2 \ 1 \ 2] = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

- 25 Suppose  $A$  and  $B$  have the same eigenvalues  $\lambda_1, \dots, \lambda_n$  with the same independent eigenvectors  $\mathbf{x}_1, \dots, \mathbf{x}_n$ . Then  $A = B$ . Reason: Any vector  $\mathbf{x}$  is a combination  $c_1\mathbf{x}_1 + \dots + c_n\mathbf{x}_n$ . What is  $A\mathbf{x}$ ? What is  $B\mathbf{x}$ ?
- 26 The block  $B$  has eigenvalues 1, 2 and  $C$  has eigenvalues 3, 4 and  $D$  has eigenvalues 5, 7. Find the eigenvalues of the 4 by 4 matrix  $A$ :

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

- 27 Find the rank and the four eigenvalues of  $A$  and  $C$ :

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

- 28 Subtract  $I$  from the previous  $A$ . Find the  $\lambda$ 's and then the determinants of

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = I - A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

- 29 (Review) Find the eigenvalues of  $A$ ,  $B$ , and  $C$ :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

- 30 When  $a + b = c + d$  show that  $(1, 1)$  is an eigenvector and find both eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- 31 If we exchange rows 1 and 2 *and* columns 1 and 2, the eigenvalues don't change. Find eigenvectors of  $A$  and  $B$  for  $\lambda = 11$ . Rank one gives  $\lambda_2 = \lambda_3 = 0$ .

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad B = PAP^T = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

- 32 Suppose  $A$  has eigenvalues 0, 3, 5 with independent eigenvectors  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$ .
- Give a basis for the nullspace and a basis for the column space.
  - Find a particular solution to  $A\mathbf{x} = \mathbf{v} + \mathbf{w}$ . Find all solutions.
  - $A\mathbf{x} = \mathbf{u}$  has no solution. If it did then \_\_\_\_\_ would be in the column space.
- 33 Suppose  $\mathbf{u}$ ,  $\mathbf{v}$  are orthonormal vectors in  $\mathbf{R}^2$ , and  $A = \mathbf{u}\mathbf{v}^T$ . Compute  $A^2 = \mathbf{u}\mathbf{v}^T\mathbf{u}\mathbf{v}^T$  to discover the eigenvalues of  $A$ . Check that the trace of  $A$  agrees with  $\lambda_1 + \lambda_2$ .
- 34 Find the eigenvalues of this permutation matrix  $P$  from  $\det(P - \lambda I) = 0$ . Which vectors are not changed by the permutation? They are eigenvectors for  $\lambda = 1$ . Can you find three more eigenvectors?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

### Challenge Problems

- 35 There are six 3 by 3 permutation matrices  $P$ . What numbers can be the *determinants* of  $P$ ? What numbers can be *pivots*? What numbers can be the *trace* of  $P$ ? What *four numbers* can be eigenvalues of  $P$ , as in Problem 15?
- 36 Is there a real 2 by 2 matrix (other than  $I$ ) with  $A^3 = I$ ? Its eigenvalues must satisfy  $\lambda^3 = 1$ . They can be  $e^{2\pi i/3}$  and  $e^{-2\pi i/3}$ . What trace and determinant would this give? Construct a rotation matrix as  $A$  (which angle of rotation?).
- 37 (a) Find the eigenvalues and eigenvectors of  $A$ . They depend on  $c$ :

$$A = \begin{bmatrix} .4 & 1 - c \\ .6 & c \end{bmatrix}.$$

- Show that  $A$  has just one line of eigenvectors when  $c = 1.6$ .
- This is a Markov matrix when  $c = .8$ . Then  $A^n$  will approach what matrix  $A^\infty$ ?