Chapter 2: The Transpose of a Derivative

Will you allow me a little calculus? This is really linear algebra for functions \( x(t) \). The matrix changes to a derivative so \( A = \frac{d}{dt} \). To find the transpose of this unusual \( A \) we need to define the inner product between two functions \( x(t) \) and \( y(t) \).

The inner product changes from the sum of \( x_k y_k \) to the integral of \( x(t) y(t) \).

\[
\text{Inner product of functions} \quad x^T y = (x, y) = \int_{-\infty}^{\infty} x(t) y(t) \, dt
\]

The transpose of a matrix has \( (Ax)^T y = x^T (A^T y) \). The “adjoint” of \( A = \frac{d}{dt} \) has

\[
(Ax, y) = \int_{-\infty}^{\infty} \frac{dx}{dt} y(t) \, dt = \int_{-\infty}^{\infty} x(t) \left(-\frac{dy}{dt}\right) \, dt = (x, A^T y)
\]

I hope you recognize integration by parts. The derivative moves from the first function \( x(t) \) to the second function \( y(t) \). During that move, a minus sign appears. This tells us that the adjoint (transpose) of the derivative is minus the derivative.

The derivative is antisymmetric: \( A = \frac{d}{dt} \) and \( A^T = -\frac{d}{dt} \). Symmetric matrices have \( S^T = S \), antisymmetric matrices have \( A^T = -A \). \( S = (d/dt)^2 \) is symmetric:

\[
A = \begin{bmatrix}
  0 & 1 & 0 & 0 \\
 -1 & 0 & 1 & 0 \\
  0 & -1 & 0 & 1 \\
  0 & 0 & -1 & 0
\end{bmatrix}
\]

transposes to

\[
A^T = \begin{bmatrix}
  0 & -1 & 0 & 0 \\
  1 & 0 & -1 & 0 \\
  0 & 1 & 0 & -1 \\
  0 & 0 & 1 & 0
\end{bmatrix} = -A.
\]

And a forward difference matrix transposes to a backward difference matrix, multiplied by \(-1\). In differential equations, the second derivative (acceleration) is symmetric. The first derivative (damping proportional to velocity) is antisymmetric.