I.1 Multiplication $Ax$ Using Columns of $A$

We hope you already know some linear algebra. It is a beautiful subject—more useful to more people than calculus (in our quiet opinion). But even old-style linear algebra courses miss basic and important facts. This first section of the book is about matrix-vector multiplication $Ax$ and the column space of a matrix.

We always use examples to make our point clear.

Example 1 Multiply $A$ times $x$ using the three rows of $A$. Then use the two columns:

**By rows**

$$
\begin{bmatrix}
2 & 3 \\
2 & 4 \\
3 & 7 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= 
\begin{bmatrix}
2x_1 + 3x_2 \\
2x_1 + 4x_2 \\
3x_1 + 7x_2 \\
\end{bmatrix}
= \text{inner products of the rows with } x = (x_1, x_2)
$$

**By columns**

$$
\begin{bmatrix}
2 & 3 \\
2 & 4 \\
3 & 7 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
= x_1 \begin{bmatrix}
2 \\
2 \\
3 \\
\end{bmatrix}
+ x_2 \begin{bmatrix}
3 \\
4 \\
7 \\
\end{bmatrix}
= \text{combination of the columns } a_1 \text{ and } a_2
$$

You see that both ways give the same result. The first way (a row at a time) produces 3 inner products. Those are also known as “dot products” because of the dot notation:

$$
\text{row} \cdot \text{column} = (2, 3) \cdot (x_1, x_2) = 2x_1 + 3x_2
$$

(1)

This is the way to find the three separate components of $Ax$. We use this for computing—but not for understanding. It is low level. Understanding is higher level, using vectors.

The vector approach sees $Ax$ as a “linear combination” of $a_1$ and $a_2$. This is the fundamental operation of linear algebra! A linear combination of $a_1$ and $a_2$ includes two steps:

1. Multiply the columns $a_1$ and $a_2$ by “scalars” $x_1$ and $x_2$
2. Add vectors $x_1a_1 + x_2a_2 = Ax$.

Thus $Ax$ is a linear combination of the columns of $A$. This is fundamental.

This thinking leads us to the column space of $A$. The key idea is to take all combinations of the columns. All real numbers $x_1$ and $x_2$ are allowed—the space includes $Ax$ for all vectors $x$. In this way we get infinitely many output vectors $Ax$. And we can see those outputs geometrically.

In our example, each $Ax$ is a vector in 3-dimensional space. That 3D space is called $\mathbb{R}^3$. (The $\mathbb{R}$ indicates real numbers. Vectors with three complex components lie in the space $\mathbb{C}^3$.) We stay with real vectors and we ask this key question:

All combinations $Ax = x_1a_1 + x_2a_2$ produce what part of the full 3D space?

Answer: Those vectors produce a plane. The plane contains the complete line in the direction of $a_1 = (2, 2, 3)$, since every vector $x_1a_1$ is included. The plane also includes the line of all vectors $x_2a_2$ in the direction of $a_2$. And it includes the sum of any vector on one line plus any vector on the other line. This addition fills out an infinite plane containing the two lines. But it does not fill out the whole 3-dimensional space $\mathbb{R}^3$. 


Definition. The combinations of the columns fill out (“span”) the column space of $A$. Here the column space is a plane. That plane includes the zero point $(0,0,0)$ which is produced when $x_1 = x_2 = 0$. The plane includes $(5,6,10) = a_1 + a_2$ and $(-1,-2,-4) = a_1 - a_2$. Every combination $x_1a_1 + x_2a_2$ is in this column space. With probability 1 it does not include the random point $b$. Which points are in the plane?

$$b = (b_1, b_2, b_3)$$ is in the column space of $A$ exactly when $Ax = b$ has a solution $(x_1, x_2)$

When you see that truth, you understand the column space $C(A)$: The solution $x$ shows how to express the right side $b$ as a combination $x_1a_1 + x_2a_2$ of the columns. For some $b$ this is impossible.

Example 2. $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not in $C(A)$. $Ax = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is unsolvable.

The first two equations force $x_1 = \frac{1}{2}$ and $x_2 = 0$. Then equation 3 fails: $3\left(\frac{1}{2}\right) + 7(0) = 1.5$ (not 1). This means that $b = (1,1,1)$ is not in the column space—the plane of $a_1$ and $a_2$.

Example 3. What are the column spaces of $A_2 = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 7 & 10 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{bmatrix}$?

Solution. The column space of $A_2$ is the same plane as before. The new column $(5,6,10)$ is the sum of column 1 + column 2. So $a_3 = \text{column 3}$ is already in the plane and adds nothing new. By including this “dependent” column we don’t go beyond the original plane.

The column space of $A_3$ is the whole 3D space $\mathbb{R}^3$. Example 2 showed us that the new third column $(1,1,1)$ is not in the plane $C(A)$. Our column space $C(A_3)$ has grown bigger. But there is nowhere to stop between a plane and the full 3D space. Visualize the $x-y$ plane and a third vector $(x_3, y_3, z_3)$ out of the plane (meaning that $z_3 \neq 0$). They combine to give every vector in $\mathbb{R}^3$.

Here is a total list of all possible column spaces inside $\mathbb{R}^3$. Dimensions 0, 1, 2, 3:

- **Subspaces of $\mathbb{R}^3$**
  - The zero vector $(0,0,0)$ by itself
  - A line of all vectors $x_1a_1$
  - A plane of all vectors $x_1a_1 + x_2a_2$
  - The whole $\mathbb{R}^3$ with all vectors $x_1a_1 + x_2a_2 + x_3a_3$

In that list we need the vectors $a_1, a_2, a_3$ to be “independent”. The only combination that gives the zero vector is $0a_1 + 0a_2 + 0a_3$. So $a_1$ by itself gives a line, $a_1$ and $a_2$ give a plane, $a_1$ and $a_2$ and $a_3$ give every vector $b$ in $\mathbb{R}^3$. The zero vector is in every subspace!

In linear algebra language:

- Three independent columns in $\mathbb{R}^3$ produce an invertible matrix: $AA^{-1} = A^{-1}A = I$.
- $Ax = 0$ requires $x = (0,0,0)$. Then $Ax = b$ has exactly one solution $x = A^{-1}b$.

You see the picture for the columns of an $n$ by $n$ invertible matrix. Their combinations fill its column space: all of $\mathbb{R}^n$. We needed those ideas and that language to go further.
Independent Columns and the Rank of $A$

After writing those words, I thought this short section was complete. Wrong. With just a small effort, we can find a basis for the column space of $A$, we can factor $A$ into $C$ times $R$, and we can prove the first great theorem in linear algebra. You will see the rank of a matrix and the dimension of a subspace.

All this comes with an understanding of independence. The goal is to create a matrix $C$ whose columns come directly from $A$—but not to include any column that is a combination of previous columns. The columns of $C$ (as many as possible) will be "independent". Here is a natural construction of $C$ from the $n$ columns of $A$:

If column 1 of $A$ is not all zero, put it into the matrix $C$.

If column 2 of $A$ is not a multiple of column 1, put it into $C$.

If column 3 of $A$ is not a combination of columns 1 and 2, put it into $C$. Continue.

At the end $C$ will have $r$ columns ($r \leq n$).

They will be a “basis” for the column space of $A$.

The left out columns are combinations of those basic columns in $C$.

A basis for a subspace is a full set of independent vectors: All vectors in the space are combinations of the basis vectors. Examples will make the point.

**Example 4** If $A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}$ then $C = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$ $n = 3$ columns in $A$ $r = 2$ columns in $C$ Column 3 of $A$ is 2 (column 1) + 2 (column 2). Leave it out of the basis in $C$.

**Example 5** If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ then $C = A$. $n = 3$ columns in $A$ $r = 3$ columns in $C$ This matrix $A$ is invertible. Its column space is all of $\mathbb{R}^3$. Keep all 3 columns.

**Example 6** If $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$ then $C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $n = 3$ columns in $A$ $r = 1$ column in $C$ The number $r$ is the “rank” of $A$. It is also the rank of $C$. It counts independent columns. Admittedly we could have moved from right to left in $A$, starting with its last column. This would not change the final count $r$. Different basis, but always the same number of vectors. That number $r$ is the “dimension” of the column space of $A$ and $C$ (same space).

The rank of a matrix is the dimension of its column space.
The matrix $C$ connects to $A$ by a third matrix $R$: $A = CR$. Their shapes are $(m \times n) = (m \times r)(r \times n)$. I can show this “factorization of $A$” in Example 4 above:

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = CR \quad (2)$$

When $C$ multiplies the first column $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ of $R$, this produces column 1 of $C$ and $A$.

When $C$ multiplies the second column $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ of $R$, we get column 2 of $C$ and $A$.

When $C$ multiplies the third column $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ of $R$, we get $2(\text{column 1}) + 2(\text{column 2})$.

This matches column 3 of $A$. All we are doing is to put the right numbers in $R$. Combinations of the columns of $C$ produce the columns of $A$. Then $A = CR$ stores this information as a matrix multiplication. Actually $R$ is a famous matrix in linear algebra:

$$R = \text{rref}(A) = \text{row-reduced echelon form of } A \text{ (without zero rows).}$$

Example 5 has $C = A$ and then $R = I$ (identity matrix). Example 6 has only one column in $C$, so it has one row in $R$:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \end{bmatrix} = CR$$

\begin{center}
All three matrices have rank $r = 1$
\end{center}

\begin{center}
\text{Column Rank } = \text{ Row Rank}
\end{center}

The number of independent columns equals the number of independent rows

This rank theorem is true for every matrix. Always columns and rows in linear algebra! The $m$ rows contain the same numbers $a_{ij}$ as the $n$ columns. But different vectors.

The theorem is proved by $A = CR$. Look at that differently—by rows instead of columns. The matrix $R$ has $r$ rows. Multiplying by $C$ takes combinations of those rows. Since $A = CR$, we get every row of $A$ from the $r$ rows of $R$. And those $r$ rows are independent, so they are a basis for the row space of $A$. The column space and row space of $A$ both have dimension $r$, with $r$ basis vectors—columns of $C$ and rows of $R$.

One minute: Why does $R$ have independent rows? Look again at Example 4.

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \leftrightarrow \text{ independent}$$

\begin{center}
\text{rows of } R
\end{center}

\begin{center}
\text{ones and zeros}
\end{center}

It is those ones and zeros in $R$ that tell me: No row is a combination of the other rows.

The big factorization for data science is the “SVD” of $A$—when the first factor $C$ has $r$ orthogonal columns and the second factor $R$ has $r$ orthogonal rows.
Problem Set I.1

1 Give an example where a combination of three nonzero vectors in \( \mathbb{R}^4 \) is the zero vector. Then write your example in the form \( A \mathbf{x} = \mathbf{0} \). What are the shapes of \( A \) and \( \mathbf{x} \) and \( \mathbf{0} \)?

2 Suppose a combination of the columns of \( A \) equals a different combination of those columns. Write that as \( A \mathbf{x} = A \mathbf{y} \). Find two combinations of the columns of \( A \) that equal the zero vector (in matrix language, find two solutions to \( A \mathbf{z} = \mathbf{0} \)).

3 (Practice with subscripts) The vectors \( a_1, a_2, \ldots, a_n \) are in \( m \)-dimensional space \( \mathbb{R}^m \), and a combination \( c_1a_1 + \cdots + c_na_n \) is the zero vector. That statement is at the vector level.

   (1) Write that statement at the matrix level. Use the matrix \( A \) with the \( a \)'s in its columns and use the column vector \( c = (c_1, \ldots, c_n) \).

   (2) Write that statement at the scalar level. Use subscripts and sigma notation to add up numbers. The column vector \( a_j \) has components \( a_{1j}, a_{2j}, \ldots, a_{mj} \).

4 Suppose \( A \) is the 3 by 3 matrix \( \text{ones}(3,3) \) of all ones. Find two independent vectors \( \mathbf{x} \) and \( \mathbf{y} \) that solve \( A \mathbf{x} = \mathbf{0} \) and \( A \mathbf{y} = \mathbf{0} \). Write that first equation \( A \mathbf{x} = \mathbf{0} \) (with numbers) as a combination of the columns of \( A \). Why don’t I ask for a third independent vector with \( A \mathbf{z} = \mathbf{0} \)?

5 The linear combinations of \( v = (1,1,0) \) and \( w = (0,1,1) \) fill a plane in \( \mathbb{R}^3 \).

   (a) Find a vector \( \mathbf{z} \) that is perpendicular to \( v \) and \( w \). Then \( \mathbf{z} \) is perpendicular to every vector \( cv + dw \) on the plane: \( (cv + dw)^T \mathbf{z} = cv^T \mathbf{z} + dw^T \mathbf{z} = 0 + 0 \).

   (b) Find a vector \( \mathbf{u} \) that is not on the plane. Check that \( \mathbf{u}^T \mathbf{z} \neq 0 \).

6 In the \( xy \) plane mark all nine of these linear combinations:

\[
c \begin{bmatrix} 2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{with} \quad c = 0, 1, 2 \quad \text{and} \quad d = 0, 1, 2.
\]

7 If three corners of a parallelogram are \((1,1), (4,2), \) and \((1,3)\), what are all three of the possible fourth corners? Draw two of them.

8 How many corners does a cube have in 4 dimensions? How many 3D faces? How many edges? An edge goes between two adjacent corners.

9 Describe the column space of \( A = [v \ w \ v + 2w] \). Describe the nullspace of \( A \): all vectors \( \mathbf{x} = (x_1, x_2, x_3) \) that solve \( A \mathbf{x} = \mathbf{0} \). Add the “dimensions” of that plane (the column space of \( A \)) and that line (the nullspace of \( A \)):

\[
\text{dimension of column space} + \text{dimension of nullspace} = \text{number of columns}
\]
10 Suppose the column space of an $m$ by $n$ matrix is all of $\mathbb{R}^3$. What can you say about $m$? What can you say about $n$? What can you say about $r$?

11 Find the matrices $C_1$ and $C_2$ containing independent columns of $A_1$ and $A_2$:

$$A_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

12 Factor each of those matrices into $A = CR$. The matrix $R$ will contain the numbers that multiply columns of $C$ to recover columns of $A$.

This is one way to look at matrix multiplication: $C$ times each column of $R$.

13 Produce a basis for the column spaces of $A_1$ and $A_2$. What are the dimensions of those column spaces—the number of independent vectors? What are the ranks of $A_1$ and $A_2$? How many independent rows in $A_1$ and $A_2$?

14 Create a 4 by 4 matrix $A$ of rank 2. What shapes are $C$ and $R$?

15 Suppose two matrices $A$ and $B$ have the same column space.

(a) Show that their row spaces can be different.

(b) Show that the matrices $C$ (basic columns) can be different.

(c) What number will be the same for $A$ and $B$?

16 IF $A = CR$, the first row of $A$ is a combination of the rows of $R$. Which part of which matrix holds the coefficients in that combination—the numbers that multiply the rows of $R$ to produce row 1 of $A$?

17 The rows of $R$ are a basis for the row space of $A$. What does that sentence mean?

18 For these matrices with square blocks, find $A = CR$. What ranks?

$$A_1 = \begin{bmatrix} \text{zeros} & \text{ones} & \text{ones} \\ \text{ones} & \text{ones} & \text{ones} \end{bmatrix}_{4 \times 4}, \quad A_2 = \begin{bmatrix} A_1 \\ A_1 \end{bmatrix}_{8 \times 4}, \quad A_3 = \begin{bmatrix} A_1 & A_1 \\ A_1 & A_1 \end{bmatrix}_{8 \times 8}$$

19 If $A = CR$, what are the $CR$ factors of the matrix $\begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$?

20 “Elimination” subtracts a number $\ell_{ij}$ times row $j$ from row $i$: a “row operation.” Show how those steps can reduce the matrix $A$ in Example 4 to $R$ (except that this row echelon form $R$ has a row of zeros). The rank won’t change!

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \rightarrow R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}(A).$$
This page is about the factorization \( A = CR \) and its close relative \( A = CMR \). As before, \( C \) has \( r \) independent columns taken from \( A \). The new matrix \( R \) has \( r \) independent rows, also taken directly from \( A \). The \( r \) by \( r \) "mixing matrix" is \( M \). This invertible matrix makes \( A = CMR \) a true equation.

The rows of \( R \) (not bold) were chosen to produce \( A = CR \), but those rows of \( R \) did not come directly from \( A \). We will see that \( R \) has the form \( MR \) (bold \( R \)).

**Rank-1 example**
\[
A = CR = CMR \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix}
\]

In this case \( M \) is just 1 by 1. How do we find \( M \) in other examples of \( A = CMR \)? \( C \) and \( R \) are not square. They have one-sided inverses. We need \( C^T C \) and \( RR^T \).

\[
A = CMR \quad C^TAR^T = C^TCM \quad RR^T = (C^T)^{-1}(C^TAR^T)(RR^T)^{-1} \quad (*)
\]

Here are extra problems to give practice with all these rectangular matrices of rank \( r \). \( C^T C \) and \( RR^T \) have rank \( r \) so they are invertible (see the last page of Section I.3).

**21** Show that equation (*) produces \( M = \begin{bmatrix} \frac{1}{2} \end{bmatrix} \) in the small example above.

**22** The rank-2 example in the text produced \( A = CR \) in equation (2):

\[
A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} = CR
\]

Choose rows 1 and 2 directly from \( A \) to go into \( R \). Then from equation (*), find the 2 by 2 matrix \( M \) that produces \( A = CMR \). Fractions enter the inverse of matrices:

**Inverse of a 2 by 2 matrix**
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (**)
\]

**23** Show that this formula (**) breaks down if \( \begin{bmatrix} b \\ d \end{bmatrix} = m \begin{bmatrix} a \\ c \end{bmatrix} \): dependent columns.

**24** Create a 3 by 2 matrix \( A \) with rank 1. Factor \( A \) into \( A = CR \) and \( A = CMR \).

**25** Create a 3 by 2 matrix \( A \) with rank 2. Factor \( A \) into \( A = CMR \).

The reason for this page is that the factorizations \( A = CR \) and \( A = CMR \) have jumped forward in importance for large matrices. When \( C \) takes columns directly from \( A \), and \( R \) takes rows directly from \( A \), those matrices preserve properties that are lost in the more famous \( QR \) and \( SVD \) factorizations. Where \( A = QR \) and \( A = U\Sigma V^T \) involve orthogonalizing the vectors, \( C \) and \( R \) keep the original data:

If \( A \) is nonnegative, so are \( C \) and \( R \). If \( A \) is sparse, so are \( C \) and \( R \).