Proof of Schur’s Theorem

David H. Wagner
dhwagnertx@mac.com

In this note, I provide more detail for the proof of Schur’s Theorem found in Strang’s
Introduction to Linear Algebra[1]

Theorem 1. If A is a square real matrix with real eigenvalues, then there is an orthogonal
matrix Q and an upper triangular matrix T such that A = QTQᵀ.

Proof. Note that A = QTQᵀ ⇔ AQ = QT. Let q₁ be an eigenvector of norm 1, with
eigenvalue λ₁. Let q₂, . . . , qₙ be any orthonormal vectors orthogonal to q₁.
Let Q₁ = [q₁, . . . , qₙ]. Then Q₁ᵀQ₁ = I, and

\[ Q₁ᵀA₁ = \begin{pmatrix} λ₁ & \cdots \\ 0 & A₂ \end{pmatrix} \]

Now I claim that A₂ has eigenvalues λ₂, . . . , λₙ. This is true because

\[
\det(A - λI) = \det(Q₁ᵀ(A - λI)Q₁) = \det(Q₁ᵀ(A - λI)Q₁)
\]

\[ = \det(Q₁ᵀAQ₁ - λQ₁ᵀQ₁) = \det \begin{pmatrix} (λ₁ - λ) & \cdots \\ 0 & (A₂ - λI) \end{pmatrix} \]

\[ = (λ₁ - λ) \det(A₂ - λI). \]

So A₂ has real eigenvalues, namely λ₂, . . . , λₙ. Now we proceed by induction. Suppose
we have proved the theorem for n = k. Then we use this fact to prove the theorem is true
for n = k + 1. Note that the theorem is trivial if n = 1.

So for n = k + 1, we proceed as above and then apply the known theorem to A₂, which
is k × k. We find that A₂ = Q₂ᵀT₂Q₂ᵀ. Now this is the hard part. Let Q₁ and A₂ be as
above, and let

\[ Q = Q₁ \begin{pmatrix} 1 & 0 \\ 0 & Q₂ \end{pmatrix} \]

Then

\[ AQ = A₁Q₁ \begin{pmatrix} 1 & 0 \\ 0 & Q₂ \end{pmatrix} = Q₁ \begin{pmatrix} λ₁ & \cdots \\ 0 & A₂ \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & Q₂ \end{pmatrix} \]

\[ = Q₁ \begin{pmatrix} λ₁ & \cdots \\ 0 & A₂Q₂ \end{pmatrix} = Q₁ \begin{pmatrix} λ₁ & \cdots \\ 0 & Q₂T₂ \end{pmatrix} \]

\[ = Q₁ \begin{pmatrix} 1 & 0 \\ 0 & Q₂ \end{pmatrix} \begin{pmatrix} λ₁ & \cdots \\ 0 & T₂ \end{pmatrix} = QT, \]

where T is upper triangular. So AQ = QT, or A = QTQᵀ.

That’s all, folks!

References