DIFFERENTIAL EQUATIONS
AND
LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

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Problem Set 7.1, page 393

1 Suppose your pulse is measured at $b_1 = 70$ beats per minute, then $b_2 = 120$, then $b_3 = 80$. The least squares solution to three equations $v = b_1, v = b_2, v = b_3$ with $A^{\mathrm{T}} = [1\ \ 1\ \ 1]$ is $\widehat{v} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} =$ _____. Use calculus and projections:

(a) Minimize $E = (v-70)^2 + (v-120)^2 + (v-80)^2$ by solving $dE/dv = 0$.

Solution (a) $\frac{dE}{dv} = 2(v-70) + 2(v-120) + 2(v-80) = 0$ at the minimizing $\widehat{v}$.

Cancel the 2's: $3v = 70 + 120 + 80 = 270$ so $\widehat{v} = v_{\text{average}} = \mathbf{90}$

(b) Project $\boldsymbol{b} = (70, 120, 80)$ onto $\boldsymbol{a} = (1,1,1)$ to find $\widehat{v} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$.

Solution (b) The projection of $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ is $\boldsymbol{p} = \boldsymbol{a}\widehat{v}$:

$$\boldsymbol{b} = \begin{bmatrix} 70 \\ 120 \\ 80 \end{bmatrix} \qquad \boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \widehat{v} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \frac{270}{3} = \mathbf{90}.$$

2 Suppose $Av = \boldsymbol{b}$ has m equations $a_i v = b_i$ in *one unknown* v. For the sum of squares $E = (a_1 v - b_1)^2 + \cdots + (a_m v - b_m)^2$, find the minimizing $\widehat{v}$ by calculus. Then form $A^{\mathrm{T}}A\widehat{v} = A^{\mathrm{T}}\boldsymbol{b}$ with one column in A, and reach the same $\widehat{v}$.

Solution To minimize E we solve $dE/dv = 0$. For $m = 3$ equations $a_i v = b_i$,

$$\frac{dE}{dv} = 2a_1(a_1 v - b_1) + 2a_2(a_2 v - b_2) + 2a_3(a_3 v - b_3) = 0 \text{ is zero when}$$

$$v = \widehat{v} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{a_1^2 + a_2^2 + a_3^2} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}.$$

When A has one column, $A^{\mathrm{T}}A\widehat{v} = A^{\mathrm{T}}\boldsymbol{b}$ is the same as $(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\widehat{v} = (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b})$.

3 With $\boldsymbol{b} = (4,1,0,1)$ at the points $x = (0,1,2,3)$ set up and solve the normal equation for the coefficients $\widehat{\boldsymbol{v}} = (C, D)$ in the nearest line $C + Dx$. Start with the four equations $A\boldsymbol{v} = \boldsymbol{b}$ that would be solvable if the points fell on a line.

Solution The unsolvable equation has $m = 4$ points on a line: only $n = 2$ unknowns.

$$A\boldsymbol{v} = \boldsymbol{b} \text{ is } \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \end{bmatrix} \text{ leading to } A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}:$$

$$\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \text{ gives } \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \frac{1}{2a} \begin{bmatrix} 60 \\ -20 \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{-1} \end{bmatrix}$$

The closest line to the four points is $\boldsymbol{b} = \mathbf{3} - \boldsymbol{x}$.

4 In Problem 3, find the projection $\boldsymbol{p} = A\boldsymbol{v}$. Check that those four values lie on the line $C + Dx$. Compute the error $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ and verify that $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$.

Solution The projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ is

$$\boldsymbol{p} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \text{ with error } \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

The best line $C + Dx = 3 - x$ does produce $\boldsymbol{p} = (3, 2, 1, 0)$ at the four points $x = 0, 1, 2, 3$.

Multiply this $\boldsymbol{e}$ by A^{T} to get $A^{\mathrm{T}}\boldsymbol{e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as expected.

5 (Problem 3 by calculus) Write down $E = ||\boldsymbol{b} - A\boldsymbol{v}||^2$ as a sum of four squares : the last one is $(1 - C - 3D)^2$. Find the derivative equations $\partial E/\partial C = \partial E/\partial D = 0$. Divide by 2 to obtain $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

Solution Minimize $E = (4 - C)^2 + (1 - C - D)^2 + (-C - 2D)^2 + (1 - C - 3D)^2$.

The partial derivatives are $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$ at the minimum :

$$-2(4 - C) - 2(1 - C - D) - 2(-C - 2D) - 2(1 - C - 3D) = 0$$

$$-2(1 - C - D) - 4(-C - 2D) - 6(1 - C - 3D) = 0$$

Factoring out -2 and collecting terms this is the same equation $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$!

$$\begin{matrix} 6 - 4C - 6D = 0 \\ 4 - 6C - 14D = 0 \end{matrix} \quad \text{or} \quad \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \widehat{C} \\ \widehat{D} \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

6 For the closest parabola $C + Dt + Et^2$ to the same four points, write down 4 unsolvable equations $A\boldsymbol{v} = \boldsymbol{b}$ for $\boldsymbol{v} = (C, D, E)$. Set up the normal equations for $\widehat{\boldsymbol{v}}$. If you fit the best cubic $C + Dt + Et^2 + Ft^3$ to those four points (thought experiment), what is the error vector $\boldsymbol{e}$?

Solution The parabola $C + Dt + Et^2$ fits the 4 points exactly if $A\boldsymbol{v} = \boldsymbol{b}$:

$$\begin{matrix} t = 0 & C + 0D + 0E = 4 \\ t = 1 & C + 1D + 1E = 1 \\ t = 2 & C + 2D + 4E = 0 \\ t = 3 & C + 3D + 9E = 1 \end{matrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}.$$

$$A^{\mathrm{T}}A = \begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \qquad .\phi A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 4+1+0+1 \\ 0+1+0+3 \\ 0+1+0+9 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}.$$

The cubic $C + Dt + Et^2 + Ft^3$ can fit 4 points exactly, with **error = zero vector**.

7 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1, b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{\boldsymbol{v}} = (C, D)$ and draw the closest line.

Solution $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

8 Find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ in Problem 7. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4)$.

Solution $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \mathbf{0}$.

9 Suppose the measurements at $t = -1, 1, 2$ are the errors $2, -6, 4$ in Problem 8. Compute $\widehat{\boldsymbol{v}}$ and the closest line to these new measurements. Explain the answer: $\boldsymbol{b} = (2, -6, 4)$ is perpendicular to _____ so the projection is $\boldsymbol{p} = \boldsymbol{0}$.

Solution If $\boldsymbol{b} =$ previous error $\boldsymbol{e}$ then $\boldsymbol{b}$ is perpendicular to the column space of A. Projection of $\boldsymbol{b}$ is $\boldsymbol{p} = \boldsymbol{0}$.

10 Suppose the measurements at $t = -1, 1, 2$ are $\boldsymbol{b} = (5, 13, 17)$. Compute $\widehat{\boldsymbol{v}}$ and the closest line $\boldsymbol{e}$. The error is $\boldsymbol{e} = \boldsymbol{0}$ because this $\boldsymbol{b}$ is _____.

Solution If $\boldsymbol{b} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ then $\widehat{\boldsymbol{x}} = (9, 4)$ and $\boldsymbol{e} = \boldsymbol{0}$ since $\boldsymbol{b}$ is *in the column space of* A.

11 Find the best line $C + Dt$ to fit $\boldsymbol{b} = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

Solution The least squares equation is $\begin{bmatrix} 5 & \mathbf{0} \\ \mathbf{0} & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$.

Solution: $C = 1$, $D = -1$. Line $1 - t$. Symmetric t's $\Rightarrow$ diagonal $A^{\mathrm{T}}A$

12 Find the *plane* that gives the best fit to the 4 values $\boldsymbol{b} = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0)$ and $(0, -1)$ of a square. At those 4 points, the equations $C + Dx + Ey = b$ are $A\boldsymbol{v} = \boldsymbol{b}$ with 3 unknowns $\boldsymbol{v} = (C, D, E)$.

Solution $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ has $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}$.

The solution $(C, D, E) = (2, -1, \frac{3}{2})$ gives the best plane $2 - x - \frac{3}{2}y$.

13 With $\boldsymbol{b} = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$ set up and solve the normal equations $A^{\mathrm{T}}A\boldsymbol{v} = A^{\mathrm{T}}\boldsymbol{b}$. For the best straight line $C + Dt$, find its four heights p_i and four errors e_i. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

Solution $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ gives $E = \|\boldsymbol{e}\|^2 = \mathbf{44}$ $\quad \widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$

14 (By calculus) Write down $E = ||\boldsymbol{b} - A\boldsymbol{v}||^2$ as a sum of four squares—the last one is $(C + 4D - 20)^2$. Find the derivative equations $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$. Divide by 2 to obtain the normal equations $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

Solution $E = (C + \mathbf{0}D)^2 + (C + \mathbf{1}D - 8)^2 + (C + \mathbf{3}D - 8)^2 + (C + \mathbf{4}D - 20)^2$. Then $\partial E/\partial C = 2C + 2(C + D - 8) + 2(C + 3D - 8) + 2(C + 4D - 20) = 0$ and $\partial E/\partial D = 1 \cdot 2(C + D - 8) + 3 \cdot 2(C + 3D - 8) + 4 \cdot 2(C + 4D - 20) = 0$.

These normal equations $\partial E/\partial C = 0$ and $\partial E/\partial D = 0$ are again $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

15 Which of the four subspaces contains the error vector $\boldsymbol{e}$? Which contains p ? Which contains $\widehat{\boldsymbol{v}}$?

Solution The error $\boldsymbol{e}$ is contained in the nullspace $\boldsymbol{N}(A^{\mathrm{T}})$, since $A^{\mathrm{T}}\boldsymbol{e} = \mathbf{0}$. The projection $\boldsymbol{p}$ is contained in the column space $\boldsymbol{C}(A)$. The vector $\widehat{\boldsymbol{v}}$ of coefficients can be any vector in $\mathbf{R}^{\boldsymbol{n}}$.

16 Find the height C of the best *horizontal line* to fit $\boldsymbol{b} = (0, 8, 8, 20)$. An exact fit would solve the four unsolvable equations $C = 0, C = 8, C = 8, C = 20$. Find the 4 by 1 matrix A in these equations and solve $A^{\mathrm{T}}A\widehat{\boldsymbol{v}} = A^{\mathrm{T}}\boldsymbol{b}$.

Solution $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$ and $A^{\mathrm{T}} = [\,1 \quad 1 \quad 1 \quad 1\,]$.

$A^{\mathrm{T}}A = [\,4\,]$. $A^{\mathrm{T}}\boldsymbol{b} = [\,36\,]$ and $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = \mathbf{9} =$ best C. $\boldsymbol{e} = (-9, -1, -1, 11)$.

17 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1, b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\widehat{\boldsymbol{v}} = (C, D)$ and draw the closest line.

Solution $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} \mathbf{9} \\ \mathbf{4} \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

18 Find the projection $\boldsymbol{p} = A\widehat{\boldsymbol{v}}$ in Problem 17. This gives the three heights of the closest line. Show that the error vector is $\boldsymbol{e} = (2, -6, 4)$. Why is $P\boldsymbol{e} = \mathbf{0}$?

Solution $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$. This error $\boldsymbol{e}$ has $P\boldsymbol{e} = P\boldsymbol{b} - P\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p} = \mathbf{0}$.

19 Suppose the measurements at $t = -1, 1, 2$ are the errors $2, -6, 4$ in Problem 18. Compute $\widehat{\boldsymbol{v}}$ and the closest line to these new measurements. Explain the answer: $\boldsymbol{b} = (2, -6, 4)$ is perpendicular to _____ so the projection is $\boldsymbol{p} = \mathbf{0}$.

Solution If $\boldsymbol{b} =$ error $\boldsymbol{e}$ then $\boldsymbol{b}$ is perpendicular to the column space of A. Projection $\boldsymbol{p} = \mathbf{0}$.

20 Suppose the measurements at $t = -1, 1, 2$ are $\boldsymbol{b} = (5, 13, 17)$. Compute $\widehat{\boldsymbol{v}}$ and the closest line and $\boldsymbol{e}$. The error is $\boldsymbol{e} = \mathbf{0}$ because this $\boldsymbol{b}$ is _____ ?

Solution If $\boldsymbol{b} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ then $\widehat{\boldsymbol{x}} = (9, 4)$ and $\boldsymbol{e} = \mathbf{0}$ since $\boldsymbol{b}$ is *in the column space of* A.

Questions 21–26 ask for projections onto lines. Also errors $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ and matrices $\boldsymbol{P}$.

21 Project the vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$. Check that $\boldsymbol{e}$ is perpendicular to $\boldsymbol{a}$:

(a) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (b) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$.

Solution (a) The projection $\boldsymbol{p}$ is

$$\boldsymbol{p} = \boldsymbol{a}\frac{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\frac{6}{3} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ perpendicular to } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution (b) In this case the projection is

$$\boldsymbol{p} = \boldsymbol{a}\frac{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}\frac{-11}{11} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

22 Draw the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ and also compute it from $\boldsymbol{p} = \widehat{v}\boldsymbol{a}$:

(a) $\boldsymbol{b} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution (a) The projection of $\boldsymbol{b} = (\cos\theta, \sin\theta)$ onto $\boldsymbol{a} = (1, 0)$ is $\boldsymbol{p} = (\cos\theta, 0)$

Solution (b) The projection of $\boldsymbol{b} = (1, 1)$ onto $\boldsymbol{a} = (1, -1)$ is $\boldsymbol{p} = (0, 0)$ since $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$.

23 In Problem 22 find the projection matrix $P = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto each vector $\boldsymbol{a}$. Verify in both cases that $P^2 = P$. Multiply $P\boldsymbol{b}$ in each case to find the projection $\boldsymbol{p}$.

Solution $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{p} = P_1\boldsymbol{b} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$. $P_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\boldsymbol{p} = P_2\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

24 Construct the projection matrices P_1 and P_2 onto the lines through the $\boldsymbol{a}$'s in Problem 22. Is it true that $(P_1 + P_2)^2 = P_1 + P_2$? This *would* be true if $P_1P_2 = 0$.

Solution The projection matrices P_1 and P_2 (note correction P_2 not $P - 2$) are

$$P_1 = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathbf{T}}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad P_2 = \frac{\boldsymbol{a}\boldsymbol{a}^{\mathbf{T}}}{\boldsymbol{a}^{\mathbf{T}}\boldsymbol{a}} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

It is *not true* that $(P_1 + P_2)^2 = P_1 + P_2$. The sum of projection matrices is **not usually** a projection matrix.

25 Compute the projection matrices $\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ onto the lines through $\boldsymbol{a}_1 = (-1, 2, 2)$ and $\boldsymbol{a}_2 = (2, 2, -1)$. Multiply those two matrices P_1P_2 and explain the answer.

Solution $P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$, $P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$.

$P_1P_2 =$ *zero matrix because* $\boldsymbol{a_1}$ is perpendicular to $\boldsymbol{a_2}$.

26 Continuing Problem 25, find the projection matrix P_3 onto $\boldsymbol{a}_3 = (2, -1, 2)$. Verify that $P_1 + P_2 + P_3 = I$. The basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ is orthogonal!

Solution $P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = I$.

We *can* add projections onto *orthogonal vectors*. This is important.

27 Project the vector $\boldsymbol{b} = (1, 1)$ onto the lines through $\boldsymbol{a}_1 = (1, 0)$ and $\boldsymbol{a}_2 = (1, 2)$. Draw the projections $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ and add $\boldsymbol{p}_1 + \boldsymbol{p}_2$. The projections do not add to $\boldsymbol{b}$ because the $\boldsymbol{a}$'s are not orthogonal.

Solution The projections of $(1, 1)$ onto the lines through $(1, 0)$ and $(1, 2)$ are $\boldsymbol{p}_1 = (1, 0)$ and $\boldsymbol{p}_2 = (3/5, 6/5) = (0.6, 1.2)$. Then $\boldsymbol{p}_1 + \boldsymbol{p}_2 \neq \boldsymbol{b}$.

28 (Quick and recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\boldsymbol{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

Solution $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $P =$ square matrix $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\boldsymbol{p} = P\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{0} \end{bmatrix}$.

29 If A is doubled, then $P = 2A(4A^{\mathrm{T}}A)^{-1}2A^{\mathrm{T}}$. This is the same as $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. The column space of $2A$ is the same as _____. Is $\widehat{\boldsymbol{v}}$ the same for A and $2A$?

Solution $2A$ has the same column space as A. Same $\boldsymbol{p}$. But $\widehat{\boldsymbol{x}}$ for $2A$ is *half* of $\widehat{\boldsymbol{x}}$ for A.

30 What linear combination of $(1,2,-1)$ and $(1,0,1)$ is closest to $\boldsymbol{b} = (2,1,1)$?

Solution $\frac{1}{2}(1,2,-1) + \frac{3}{2}(1,0,1) = (2,1,1)$. So $\boldsymbol{b}$ is in the plane: no error $\boldsymbol{e}$. Projection shows $P\boldsymbol{b} = \boldsymbol{b}$.

31 (*Important*) If $P^2 = P$ show that $(I-P)^2 = I-P$. When P projects onto the column space of A, $I-P$ projects onto which fundamental subspace ?

Solution If $P^2 = P$ then $(\boldsymbol{I}-\boldsymbol{P})^{\mathbf{2}} = (I-P)(I-P) = I-PI-IP+P^2 = \boldsymbol{I}-\boldsymbol{P}$. When P projects onto the column space, $I-P$ projects onto the *left nullspace*.

32 If P is the 3 by 3 projection matrix onto the line through $(1,1,1)$, then $I-P$ is the projection matrix onto _____.

Solution $I-P$ is the projection onto the plane $x_1+x_2+x_3 = 0$, perpendicular to the direction $(1,1,1)$:

$$I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

33 Multiply the matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ by itself. Cancel to prove that $P^2 = P$. Explain why $P(P\boldsymbol{b})$ always equals $P\boldsymbol{b}$: The vector $P\boldsymbol{b}$ is in the column space so its projection is _____.

Solution $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2 = A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. So $P^2 = P$. Geometric reason: $P\boldsymbol{b}$ is in the column space (where P projects). Then its projection $P(P\boldsymbol{b})$ is $P\boldsymbol{b}$ for every $\boldsymbol{b}$. So $P^2 = P$.

34 If A is square and invertible, the warning against splitting $(A^{\mathrm{T}}A)^{-1}$ does not apply. Then $AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$ is true. *When A is invertible, why is $P = I$* and $\boldsymbol{e} = \boldsymbol{0}$?

Solution If A is invertible then its column space is all of $\mathbf{R}^n$. So $P = I$ and $\boldsymbol{e} = \mathbf{0}$.

35 An important fact about $A^{\mathrm{T}}A$ is this: ***If*** $\boldsymbol{A}^{\mathbf{T}}\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ ***then*** $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$. *New proof*: The vector $A\boldsymbol{x}$ is in the nullspace of _____. $A\boldsymbol{x}$ is always in the column space of _____. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero.

Solution If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ is in the *nullspace of* A^{T}. But $A\boldsymbol{x}$ is always in the *column space of* A. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero. So A and $A^{\mathrm{T}}A$ have the *same nullspace*.

Notes on mean and variance and test grades

If all grades on a test are 90, the mean is $m = 90$ and the variance is $\sigma^2 = 0$. Suppose the expected grades are $g_1, \ldots, g_N$. Then σ^2 comes from *squaring distances to the mean*:

$$\textbf{Mean} \quad m = \frac{g_1 + \cdots + g_N}{N} \qquad \textbf{Variance} \quad \sigma^2 = \frac{(g_1 - m)^2 + \cdots + (g_N - m)^2}{N}$$

After every test my class wants to know m and σ. My expectations are usually way off.

36 Show that σ^2 also equals $\frac{1}{N}(g_1^2 + \cdots + g_N^2) - m^2$.

Solution Each term $(g_i - m)^2$ equals $g_i^2 - 2g_i m + m^2$, so

$$\begin{aligned}\sigma^2 &= \frac{(\text{sum of } g_i^2) - 2m(\text{sum of } g_i) + Nm^2}{N} = \frac{(\text{sum of } g_i^2) - 2mNm + Nm^2}{N} \\ &= \frac{(\text{sum of } g_i^2)}{N} - m^2.\end{aligned}$$

37 If you flip a fair coin N times (1 for heads, 0 for tails) what is the expected number m of heads ? What is the variance σ^2 ?

Solution For a fair coin you expect $\boldsymbol{N/2}$ heads in N flips. The variance σ^2 turns out to be $\boldsymbol{N/4}$.

Problem Set 7.4, page 422

1 What solution to Laplace's equation completes "degree 3" in the table of pairs of solutions ? We have one solution $u = x^3 - 3xy^2$, and we need another solution.

Solution Start with $s = -y^3$. Then $s_{yy} = -6y$, and therefore we need $s_{xx} = 6y$. Integrating twice with respect to x gives $3y^2x$. Therefore the second function is $s(x, y) = -y^3 + 3x^2y$.

2 What are the two solutions of degree 4, the real and imaginary parts of $(x + iy)^4$? Check $u_{xx} + u_{yy} = 0$ for both solutions.

Solution Expanding $(x + iy)^4$ gives

$$(x + iy)^4 = x^4 - 6x^2y^2 + y^4 + (4x^3y - 4xy^3)i$$

Therefore the two solutions would be :

$$u(x, y) = \boldsymbol{x^4 - 6x^2y^2 + y^4} \quad \text{and} \quad s(x, y) = \boldsymbol{4x^3y - 4xy^3}$$

Checking the first solution :

$$\frac{\partial^2(x^4 - 6x^2y^2 + y^4)}{\partial x^2} + \frac{\partial^2(x^4 - 6x^2y^2 + y^4)}{\partial y^2} = (12x^2 - 12y^2) + (-12x^2 + 12y^2) = 0$$

Checking the second solution :

$$\frac{\partial^2(4x^3y - 4xy^3)}{\partial x^2} + \frac{\partial^2(4x^3y - 4xy^3)}{\partial y^2} = (24xy - 0) + (0 - 24xy) = 0$$

3 What is the second x-derivative of $(x + iy)^n$? What is the second y-derivative ? Those cancel in $u_{xx} + u_{yy}$ because $i^2 = -1$.

Solution The second x-derivative of $(x + iy)^n$ is :

$$\frac{\partial^2(x + iy)^n}{\partial x^2} = n(n - 1)(x + iy)^{n-2}$$

The second y-derivative of $(x + iy)^n$ cancels that because

$$\frac{\partial^2(x + iy)^n}{\partial y^2} = i \cdot i \cdot n(n - 1)(x + iy)^{n-2} = -n(n - 1)(x + iy)^{n-2}$$

4 For the solved 2×2 example inside a 4×4 square grid, write the four equations (9) at the four interior nodes. Move the known boundary values 0 and 4 to the right hand sides of the equations. You should see $K2\mathrm{D}$ on the left side multiplying the correct solution $\boldsymbol{U} = (U_{11}, U_{12}, U_{21}, U_{22}) = (1, 2, 2, 3)$.

Solution The equations at the interior node would be:

$$\begin{aligned}
4U_{1,1} - U_{2,1} - U_{0,1} - U_{1,2} - U_{1,0} &= 0\\
4U_{1,2} - U_{2,2} - U_{0,2} - U_{1,3} - U_{1,1} &= 0\\
4U_{2,1} - U_{3,1} - U_{1,1} - U_{2,2} - U_{2,0} &= 0\\
4U_{2,2} - U_{3,2} - U_{1,2} - U_{2,3} - U_{2,1} &= 0
\end{aligned}$$

Substituting the known boundary values leaves:

$$\begin{aligned}
4U_{1,1} - U_{2,1} - U_{1,2} &= 4\\
4U_{1,2} - U_{2,2} - U_{1,1} &= 8\\
4U_{2,1} - U_{1,1} - U_{2,2} &= 0\\
4U_{2,2} - U_{1,2} - U_{2,1} &= 4
\end{aligned}$$

Writing this in matrix form gives:

$$\begin{bmatrix} 4 & -1 & 0 & -1\\ -1 & 4 & -1 & 0\\ 0 & -1 & 4 & -1\\ -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} U_{1,1}\\ U_{1,2}\\ U_{2,1}\\ U_{2,2} \end{bmatrix} = \begin{bmatrix} 4\\ 8\\ 0\\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U_{1,1}\\ U_{1,2}\\ U_{2,1}\\ U_{2,2} \end{bmatrix} = \begin{bmatrix} 2\\ 3\\ 1\\ 2 \end{bmatrix}$$

5 Suppose the boundary values on the 4×4 grid change to $U = 0$ on three sides and $U = 8$ on the fourth side. Find the four inside values so that each one is the average of its neighbors.

Solution The values at the 16 nodes will be

$$\begin{matrix} 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{3}{2} & \frac{3}{2} & 0\\ 0/4 & 4 & 4 & 0/4 \end{matrix}$$

Notice that the corner boundary values **do not enter** the 5-point equations around interior points. Every interior value must be the average of its four neighbors. By symmetry the two middle columns must be the same.

6 (MATLAB) Find the inverse $(K2\mathrm{D})^{-1}$ of the 4 by 4 matrix displayed for the square grid.

Solution The circulant matrix $K2\mathrm{D}$ on page 422 has a circulant inverse:

$$(K2\mathrm{D})^{-1} = \frac{1}{24} \begin{bmatrix} 7 & 2 & 1 & 2\\ 2 & 7 & 2 & 1\\ 1 & 2 & 7 & 2\\ 2 & 1 & 2 & 7 \end{bmatrix}.$$

7 Solve this Poisson finite difference equation (right side $\neq 0$) for the inside values $U_{11}, U_{12}, U_{21}, U_{22}$. All boundary values like U_{10} and U_{13} are zero. The boundary has i or j equal to 0 or 3, the interior has i *and* j equal to 1 or 2:

$$4U_{ij} - U_{i-1,j} - U_{i+1,j} - U_{i,j-1} - U_{i,j+1} = \mathbf{1} \text{ at four inside points.}$$

Solution The interior solution to the Poisson equation (on this small grid) is

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

On a larger grid U_{ij} will not be constant in the interior.

8 A 5×5 grid has a 3 by 3 interior grid: 9 unknown values U_{11} to U_{33}. Create the 9×9 difference matrix $K2$D.

Solution Order the points by rows to get $U_{11}, U_{12}, U_{13}, U_{21}, U_{22}, U_{23}, U_{31}, U_{32}, U_{33}$. Then $K2D$ is symmetric with 3 by 3 blocks:

$$K2D = \begin{bmatrix} A & -I & 0 \\ -I & A & -I \\ 0 & -I & A \end{bmatrix} \qquad A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

9 Use eig$(K2\text{D})$ to find the nine eigenvalues of $K2$D in Problem 8. Those eigenvalues will be positive! The matrix $K2$D is symmetric positive definite.

Solution eig$(K2\text{D})$ in Problem 8 produces 9 eigenvalues between 0 and 4:

The eigenvalues come from eig$(K2\text{D})$ and explicitly from equation (11). Notice that pairs of eigenvalues add to 8. The eigenvalue distribution is symmetric around $\lambda = 4$:

$$1.1716 \quad 2.5828 \quad 2.5828 \quad 4.0 \quad 4.0 \quad 4.0 \quad 5.4142 \quad 5.4142 \quad 6.8284$$

10 If $u(x)$ solves $u_{xx} = 0$ and $v(y)$ solves $v_{yy} = 0$, verify that $u(x)v(y)$ solves Laplace's equation. Why is this only a 4-dimensional space of solutions? Separation of variables does not give all solutions—only the solutions with separable boundary conditions.

Solution

$$\text{If } \frac{\partial^2 u}{\partial x^2} = 0 \text{ and } \frac{\partial^2 v}{\partial y^2} = 0 \text{ then}$$

$$\begin{aligned} \frac{\partial^2 u(x)v(y)}{\partial x^2} + \frac{\partial^2 u(x)v(y)}{\partial y^2} &= v(y)\frac{\partial^2 u(x)}{\partial x^2} + u(x)\frac{\partial^2 v(y)}{\partial y^2} \\ &= v \cdot 0 + u \cdot 0 = 0 \end{aligned}$$

Therefore $u(x)v(y)$ solves Laplace's equation. But the only solutions found this way are $u(x)v(y) = (A + Bx)(C + Dy)$.

Problem Set 7.5, page 428

Problems 1 − 5 are about complete graphs. Every pair of nodes has an edge.

1 With $n = 5$ nodes and all edges, find the diagonal entries of $A^{\mathrm{T}}A$ (the degrees of the nodes). All the off-diagonal entries of $A^{\mathrm{T}}A$ are -1. Show the reduced matrix R without row 5 and column 5. Node 5 is "grounded" and $v_5 = 0$.

Solution The complete graph (all edges included) has no zeros in $A^{\mathrm{T}}A$:

$$A^{\mathrm{T}}A = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} \qquad \textbf{Singular!}$$

The grounded matrix would be

$$(A^{\mathrm{T}}A)_{\text{reduced}} = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \qquad \textbf{Invetible!}$$

2 Show that the *trace* of $A^{\mathrm{T}}A$ (sum down the diagonal = sum of eigenvalues) is $n^2 - n$. What is the trace of the reduced (and invertible) matrix R of size $n-1$?

Solution $A^{\mathrm{T}}A$ is n by n and each diagonal entry is $n-1$. Therefore the trace is $n(n-1) = \boldsymbol{n^2 - n}$. The reduced matrix R has $n-1$ diagonal entries, each still equal to $n-1$. Therefore the trace is $(n-1)(n-1) = n^2 - 2n + 1$.

3 For $n = 4$, write the 3 by 3 matrix $R = (A_{\text{reduced}})^{\mathrm{T}}(A_{\text{reduced}})$. Show that $RR^{-1} = I$ when R^{-1} has all entries $\frac{1}{4}$ off the diagonal and $\frac{2}{4}$ on the diagonal.

Solution

$$\textbf{Reduced matrix } R = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

R by its proposed inverse gives

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

4 For every n, the reduced matrix R of size $n-1$ is *invertible*. Show that $RR^{-1} = I$ when R^{-1} has all entries $1/n$ off the diagonal and $2/n$ on the diagonal.

Solution

$$\frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-1-1 & 3-2-1 & 3-1-2 \\ -2+3-1 & -1+6-1 & -1+3-2 \\ -2-1+3 & -1-2+3 & -1-1+6 \end{bmatrix} = I.$$

5 Write the 6 by 3 matrix $M = A_{\text{reduced}}$ when $n = 4$. The equation $M\boldsymbol{v} = \boldsymbol{b}$ is to be solved by least squares. The vector $\boldsymbol{b}$ is like scores in 6 games between 4 teams (team 4 always scores zero; it is grounded). Knowing the inverse of $R = M^{\mathrm{T}}M$, what is the least squares ranking $\widehat{v}_1$ for team 1 from solving $M^{\mathrm{T}}M\widehat{\boldsymbol{v}} = M^{\mathrm{T}}\boldsymbol{b}$?

Solution Remove column 4 of A when node 4 is grounded ($x_4 = 0$).

$$M = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ has independent columns}$$

The least squares solution $\widehat{v}$ to $M\boldsymbol{v} = \boldsymbol{b}$ comes from $M^{\mathrm{T}}M\widehat{\boldsymbol{v}} = M^{\mathrm{T}}\boldsymbol{b}$. This $\widehat{\boldsymbol{v}}$ gives the predicted point spreads when all teams play all other teams. The first component $\widehat{v_1}$ would come from the first row of $(M^{\mathrm{T}}M)^{-1}$ multiplying by $M^{\mathrm{T}}\boldsymbol{b}$. Note that

$$M^{\mathrm{T}}M = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \text{ and } (M^{\mathrm{T}}M)^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

6 For the tree graph with 4 nodes, $A^{\mathrm{T}}A$ is in equation (1). What is the 3 by 3 matrix $R = (A^{\mathrm{T}}A)_{\text{reduced}}$? How do we know it is positive definite?

Solution The reduced form of $A^{\mathrm{T}}A$ removes row 4 and column 4:

$$\text{Singular } A^{\mathrm{T}}A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \text{ reduces to invertible } \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The first is positive semidefinite (A has dependent columns). the second is positive definite (the reduced A has 3 independent columns).

7 (a) If you are given the matrix A, how could you reconstruct the graph?

Solution Each row of A tells you an edge in the graph.

(b) If you are given $L = A^{\mathrm{T}}A$, how could you reconstruct the graph (no arrows)?

Solution Each nonzero off the main diagonal of $A^{\mathrm{T}}A$ tells you an edge.

(c) If you are given $K = A^{\mathrm{T}}CA$, how could you reconstruct the weighted graph?

Solution Each nonzero off the main diagonal tells you the weight of that edge.

8 Find $K = A^{\mathrm{T}}CA$ for a line of 3 resistors with conductances $c_1 = 1$, $c_2 = 4$, $c_3 = 9$. Write K_{reduced} and show that this matrix is positive definite.

Solution A **circle** of three resistors has 3 edges and 3 nodes:

$$\begin{aligned} A^{\mathrm{T}}CA &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & -4 & -1 \\ -4 & 13 & -9 \\ -1 & -9 & 10 \end{bmatrix} \text{ is only } \textbf{semidefinite} \\ (A^{\mathrm{T}}CA)_{\text{reduced}} &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 4 & \\ & & 9 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{-4} \\ \mathbf{-4} & \mathbf{13} \end{bmatrix} \end{aligned}$$

The determinant tests $5 > 0$ and $(5)(13) > 4^2$ are passed.

9 A 3 by 3 square grid has $n = 9$ nodes and $m = 12$ edges. Number nodes by rows.

(a) How many nonzeros among the 81 entries of $L = A^{\mathrm{T}}A$?

Solution The 9 nodes ordered by rows have $2, 3, 2, 3, 4, 3, 2, 3, 2$ neighbors around them. Those add to 24 nonzeros off the diagonal. The 9 diagonal entries make 33 nonzeros out of $9^2 = 81$ entries in $L = A^{\mathrm{T}}A$.

(b) Write down the 9 diagonal entries in the degree matrix D: they are not all 4.

Solution Those 9 numbers are the degrees of the 9 nodes ($=$ diagonal entries in $A^{\mathrm{T}}A$).

(c) Why does the middle row of $L = D - W$ have four -1's? Notice $L = K2\mathrm{D}$!

Solution The middle node in the grid has **4 neighbors**.

10 Suppose all conductances in equation (5) are equal to c. Solve equation (6) for the voltages v_2 and v_3 and find the current I flowing out of node 1 (and into the ground at node 4). What is the "system conductance" I/V from node 1 to node 4?

This overall conductance I/V should be larger than the individual conductances c.

Solution The reduced equation (6) with conductances $= c$ is

$$\begin{bmatrix} 3c & -c \\ -c & 2c \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} cV \\ cV \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0.6V \\ 0.8V \end{bmatrix}.$$

Then the flows on the five edges in Figure 7.6 use A in equation (2). Remember the minus sign:

$$-cA\boldsymbol{v} = -c \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} V \\ 0.6V \\ 0.8V \\ 0 \end{bmatrix} = cV \begin{bmatrix} 0.4 \\ 0.2 \\ -0.2 \\ 1.0 \\ 0.6 \end{bmatrix}$$

The total flow (on edges $1{+}2{+}4$ out of node 1, or on edges $3{+}4$ into the grounded node 4, is $I = 1.6cV$. The overall system conductance is $1.6c$, greater than the individual conductance c on each edge.

11 The multiplication $A^{\mathrm{T}}A$ can be columns of A^{T} times rows of A. For the tree with $m = 3$ edges and $n = 4$ nodes, each (column times row) is $(4 \times 1)(1 \times 4) = 4 \times 4$. Write down those three column-times-row matrices and add to get $L = A^{\mathrm{T}}A$.

Solution Suppose the 3 tree edges go out of node 1 to nodes $2, 3, 4$. (The problem allows to choose other trees, including a line of 4 nodes.) Then

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad A^{\mathrm{T}}A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \text{ sum of (columns of} A^{\mathrm{T}})(\text{ rows of } A)$$

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}.$$

12 A graph with two separate 3-node trees is *not connected*. Write its 6 by 4 incidence matrix A. Find *two* solutions to $A\boldsymbol{v} = \mathbf{0}$, not just one solution $\boldsymbol{v} = (1, 1, 1, 1, 1, 1)$. To reduce $A^{\mathrm{T}}A$ we must ground *two* nodes and remove two rows and columns.

Solution The incidence matrix for two 3-node trees is

$$A = \begin{bmatrix} A_{\text{tree}} & 0 \\ 0 & A_{\text{tree}} \end{bmatrix} \text{ with } A_{\text{tree}} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ (for example)}$$

The columns of A_{tree} add to zero so we have 2 independent solutions to $A\boldsymbol{v} = \mathbf{0}$:

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ come from } A_{\text{tree}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

13 "Element matrices" from column times row appear in the **finite element method**. Include the numbers c_1, c_2, c_3 in the element matrices K_1, K_1, K_3.

$$K_i = (\text{row } i \text{ of } A)^{\mathrm{T}} \;\; (\boldsymbol{c_i}) \;\; (\text{row } i \text{ of } A) \qquad K = A^{\mathrm{T}}CA = \boldsymbol{K_1 + K_2 + K_3}.$$

Write the element matrices that add to $A^{\mathrm{T}}A$ in (1) for the 4-node line graph.

$$A^{\mathrm{T}}A = \begin{bmatrix} K_1 & & \\ & K_2 & \\ & & K_3 \end{bmatrix} = \begin{array}{l} \text{assembly of the nonzero} \\ \text{entries of } K_1 + K_2 + K_3 \\ \text{from edges } 1, 2, \text{ and } 3 \end{array}$$

Solution The three "element matrices" for the three edges come from multiplying the three columns of A^{T} by the three rows of A. Then $A^{\mathrm{T}}A$ equals

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}.$$

When the diagonal matrix C is included, those are multiplied by c_1, c_2, and c_3. Those products produce 2 by 2 blocks of nonzeros in 4×4 matrices:

$$K_1 = c_1 \begin{bmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & & \\ & & & \end{bmatrix} \quad K_2 = c_2 \begin{bmatrix} & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & \end{bmatrix} \quad K_3 = c_3 \begin{bmatrix} & & & \\ & & & \\ & & 1 & -1 \\ & & -1 & 1 \end{bmatrix}$$

Then $\boldsymbol{A^{\mathrm{T}}CA = K_1 + K_2 + K_3}$. This 'assembly" of the element stiffness matrices just requires placing the nonzeros correctly into the final matrix $A^{\mathrm{T}}CA$.

14 An n by n grid has n^2 nodes. How many edges in this graph? How many interior nodes? How many nonzeros in A and in $L = A^{\mathrm{T}}A$? *There are no zeros in* L^{-1}!

Solution An n by n grid has n horizontal rows ($n-1$ edges on each row) and n vertical columns ($n - 1$ edges down each column). Altogether $\boldsymbol{2n(n-1)}$ edges. There are

$\boldsymbol{(n-2)^2}$ interior nodes—a square grid with the boundary nodes removed to reduce n to $n-2$.

Every edge produces 2 nonzeros (-1 and $+1$) in A. Then A has $\boldsymbol{4n(n-1)}$ nonzeros. The matrix $A^{\mathrm{T}}A$ has size n^2 with n^2 diagonal nonzeros—and off the diagonal of $A^{\mathrm{T}}A$ there are two -1's for each edge: altogether $n^2+4n(n-1)=\boldsymbol{5n^2-4n}$ nonzeros out of n^4 entries. For $n=2$, this means 12 nonzeros in a 4 by 4 matrix.

15 When only $\boldsymbol{e}=C^{-1}\boldsymbol{w}$ is eliminated from the 3-step framework, equation (**??**) shows

$$\textbf{Saddle-point matrix}\quad \textbf{Not positive definite}\qquad \begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}.$$

Multiply the first block row by $A^{\mathrm{T}}C$ and subtract from the second block row:

$$\textbf{After block elimination}\qquad \begin{bmatrix} C^{-1} & A \\ 0 & -A^{\mathrm{T}}CA \end{bmatrix}\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f}-A^{\mathrm{T}}C\boldsymbol{b} \end{bmatrix}.$$

After m positive pivots from C^{-1}, why does this matrix have negative pivots? The two-field problem for $\boldsymbol{w}$ and $\boldsymbol{v}$ is finding a saddle point, not a minimum.

Solution The three equations $e=b-Av$ and $w=Ce$ and $A^{\mathrm{T}}w=f$ reduce to two equations when e is replaced by $C^{-1}w$:

$$\begin{aligned} C^{-1}\boldsymbol{w}&=\boldsymbol{b}-Av \\ A^{\mathrm{T}}\boldsymbol{w}&=\boldsymbol{f} \end{aligned}\qquad \text{become} \qquad \begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f} \end{bmatrix}.$$

Multiply the first equation by $A^{\mathrm{T}}C$ to get $A^{\mathrm{T}}\boldsymbol{w}=A^{\mathrm{T}}C\boldsymbol{b}-A^{\mathrm{T}}CA\boldsymbol{v}$. Subtract from the second equation $A^{\mathrm{T}}\boldsymbol{w}=\boldsymbol{f}$, to eliminate w:

$$A^{\mathrm{T}}C\boldsymbol{b}-A^{\mathrm{T}}CA\boldsymbol{v}=\boldsymbol{f}.$$

This gives the second row of the block matrix after elimination:

$$\begin{bmatrix} C^{-1} & A \\ 0 & -A^{\mathrm{T}}CA \end{bmatrix}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}=\begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{f}-A^{\mathrm{T}}C\boldsymbol{b} \end{bmatrix}.$$

The pivots of that matrix on the left side start with $1/c_1,1/c_2,\ldots,1/c_m$. Then we get the n pivots of $-A^{\mathrm{T}}CA$ which are **negative**, because this matrix is negative definite.

Altogether we are finding a saddle point $(\boldsymbol{v},\boldsymbol{w})$ of the energy (quadratic function). The derivative of that quadratic gives our linear equations. The block matrix in those equations has m positive eigenvalues and n negative eigenvalues.

16 The least squares equation $A^{\mathrm{T}}A\boldsymbol{v}=A^{\mathrm{T}}\boldsymbol{b}$ comes from the projection equation $A^{\mathrm{T}}\boldsymbol{e}=\boldsymbol{0}$ for the error $\boldsymbol{e}=\boldsymbol{b}-A\boldsymbol{v}$. Write those two equations in the symmetric saddle point form of Problem 7 (with $\boldsymbol{f}=\boldsymbol{0}$).

In this case $\boldsymbol{w}=\boldsymbol{e}$ because the weighting matrix is $C=I$.

Solution Ordinary least squares for $A\boldsymbol{v}=\boldsymbol{b}$ separates the data vector $\boldsymbol{b}$ in two perpendicular parts:

$$\boldsymbol{b}=(A\widehat{\boldsymbol{v}})+(\boldsymbol{b}-A\widehat{\boldsymbol{v}})=(\text{projection of }\boldsymbol{b})+(\text{error in }\boldsymbol{b}).$$

The error $\boldsymbol{e}=\boldsymbol{b}-A\boldsymbol{v}$ satisfies $A^{\mathrm{T}}\boldsymbol{e}=A^{\mathrm{T}}\boldsymbol{b}-A^{\mathrm{T}}A\boldsymbol{v}=\boldsymbol{0}$ (which means that $A^{\mathrm{T}}A\boldsymbol{v}=A^{\mathrm{T}}\boldsymbol{b}$, the key equation). That equation $d^{\mathrm{T}}\boldsymbol{e}=\boldsymbol{0}$ is Kirchhoff's Current Law for flows in a network. It is a candidate for the "most important equation in applied mathematics"—the conservation equation or continuity equation "flow in $=$ flow out."

In the form of Problem 15 (with $C = I$) the equations are

$$\begin{bmatrix} I & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} \quad \text{or} \quad \begin{array}{ll} \boldsymbol{e} + A\boldsymbol{v} = \boldsymbol{b} \\ A^{\mathrm{T}}\boldsymbol{e} \quad\; = \boldsymbol{0}\,. \end{array}$$

17 Find the three eigenvalues and three pivots and the determinant of this saddle point matrix with $C = I$. One eigenvalue is negative because A has one column:

$$m = 2, n = 1 \qquad \begin{bmatrix} C^{-1} & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Solution The eigenvalues come from $\det(M - \lambda I) = 0$:

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 1 \\ -1 & 1 & -\lambda \end{bmatrix} = -\lambda(1-\lambda)^2 - 2(1-\lambda) = 0\,.$$

Then $(1-\lambda)(\lambda^2 - \lambda - 2) = 0$ and $(1-\lambda)(\lambda - 2)(\lambda + 1) = 0$ and the eigenvalues are $\lambda = \mathbf{1}, \mathbf{2}, \mathbf{-1}$. Check the sum $1 + 2 - 1 = \mathbf{2}$ equal to the trace (sum down the main diagonal $1 + 1 + 0 = \mathbf{2}$).

The determinant is the product $\lambda_1\lambda_2\lambda_3 = (1)(2)(-1) = -2$. Notice $m = 2$ positive λ's and $n = 1$ negative eigenvalue.

Elimination finds the three pivots (which also multiply to give $\det M = -2$):

$$\begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 1 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 1 \\ 0 & 0 & \textcircled{-2} \end{bmatrix}.$$

Problem Set 8.1, page 443

1 (a) To prove that cos nx is orthogonal to cos kx when $k \neq n$, use $(\cos nx)\ (\cos kx) = \frac{1}{2}\cos(n+k)x + \frac{1}{2}\cos(n-k)x$. Integrate from $x = 0$ to $x = \pi$. What is $\int \cos^2 kx\, dx$?

(b) **Correction** From 0 to π, **cos x is not orthogonal to sin $2x$** (the book wrongly proposed $\int_0^\pi \cos x\ \sin x\, dx$, but this is zero). For orthogonality of **all** sines and cosines, the period has to be 2π.

Solution (a)

$$\int_0^\pi (\cos nx)(\cos kx)dx = \frac{1}{2}\int_0^\pi \cos(n+k)x\,dx + \frac{1}{2}\int_0^\pi \cos(n-k)x\,dx$$

$$= \left[\frac{\sin(n+k)x}{2(n+k)} + \frac{\sin(n-k)x}{2(n-k)}\right]_0^\pi = 0 + 0$$

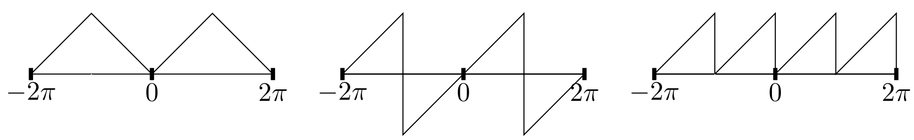
Solution (b) $$\int_0^\pi (\cos x)(\sin 2x)\,dx = \int_0^\pi (\cos x)(2\sin x\cos x)\,dx = \left[-\frac{2}{3}\cos^3 x\right]_0^\pi$$

$$= \mathbf{\frac{4}{3}} \neq 0.$$

Non-orthogonality comes from $\int_0^\pi \cos mx\ \sin nx\, dx$ when $m - n$ is an odd number.

2 Suppose $F(x) = x$ for $0 \leq x \leq \pi$. Draw graphs for $-2\pi \leq x \leq 2\pi$ to show three extensions of F: a 2π-periodic even function and a 2π-periodic odd function and a π-periodic function.

Solution



3 Find the Fourier series on $-\pi \leq x \leq \pi$ for

(a) $f_1(x) = \sin^3 x$, an odd function (sine series, only two terms)

Solution (a) The fast way is to know the identity $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$. This must be the Fourier sine series! It has only two terms.

More slowly, use Euler's great formula to produce complex exponentials :

$$(\sin x)^3 = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 = \frac{e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}}{8i^3} = -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x.$$

Or slowly compute the usual formulas $\int \sin^3 x \sin x\, dx$ and $\int \sin^3 x \sin 3x\, dx$.

(b) $f_2(x) = |\sin x|$, an even function (cosine series)

Solution (b)

$$a_0 = \frac{1}{\pi}\int_0^{\pi} |\sin x|\, dx = \frac{2}{\pi}$$

$$a_k = \frac{1}{2\pi}\int_0^{\pi} |\sin x|\, \cos\, kx\, dx = -\frac{1}{4\pi}\left[\frac{\cos(k-1)x}{k-1} + \frac{\cos(k+1)x}{k+1}\right]_{x=0}^{x=\pi}$$

$$= \mathbf{0}\ (\text{odd } k) \text{ or } -\frac{1}{4\pi}\left[\frac{-2}{k-1} + \frac{-2}{k+1}\right] = \boldsymbol{\frac{k}{\pi(k^2-1)}}\ (\text{even } k)$$

(c) $f_3(x) = x$ for $-\pi \le x \le \pi$ (sine series with jump at $x = \pi$)

Solution (c) $b_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x\, \sin\, kx\, dx = \left[\frac{1}{\pi\, k^2}\sin\, kx - \frac{x}{\pi\, k}\cos\, kx\right]_{-\pi}^{\pi}$

$$= -\frac{1}{k}(\cos\, k\pi + \cos(-k\pi)) = -\frac{2}{k}(-1)^k.$$

4 Find the complex Fourier series $e^x = \sum c_k e^{ikx}$ on the interval $-\pi \le x \le \pi$. The even part of a function is $\frac{1}{2}(f(x) + f(-x))$, so that $f_{\text{even}}(x) = f_{\text{even}}(-x)$. Find the cosine series for f_{even} and the sine series for f_{odd}. *Notice the jump at* $x = \pi$.

Solution

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ikx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-ik)}\, dx$$

$$= \left[\frac{1}{2\pi(1-ik)} e^{x(1-ik)}\right]_{-\pi}^{\pi} = \frac{e^{\pi(1-ik)} - e^{-\pi(1-ik)}}{2\pi(1-ik)}$$

The even part of the function is: $\dfrac{1}{2}(e^x + e^{-x})$. The cosine coefficients are

$$a_0 = \frac{1}{4\pi}\int_{-\pi}^{\pi} (e^x + e^{-x})\, dx = \frac{1}{2\pi}(e^{\pi} - e^{-\pi})$$

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} (e^x + e^{-x}) \cos kx\, dx = \frac{2k\cosh[\pi]\sin[k\pi] + 2\cos[k\pi]\sinh[\pi]}{\pi + k^2\pi}$$

The odd part of the function is: $\dfrac{1}{2}(e^x - e^{-x})$. The sine series is:

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} (e^x - e^{-x}) \sin kx\, dx = \frac{2\cosh[\pi]\sin[k\pi] - 2k\cos[k\pi]\sinh[\pi]}{\pi + k^2\pi}$$

5 From the energy formula (21), the square wave sine coefficients satisfy

$$\pi(b_1^2 + b_2^2 + \cdots) = \int_{-\pi}^{\pi} |SW(x)|^2\, dx = \int_{-\pi}^{\pi} 1\, dx = 2\pi.$$

Substitute the numbers b_k from equation (8) to find that $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \cdots)$.

Solution The sine coefficients for the odd square wave are

$$b_k = \frac{4}{\pi}\left(\frac{1-(-1)^k}{2k}\right) = \frac{4}{\pi}\left(\frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \ldots\right)$$

Energy identity gives $\boldsymbol{\pi^2} = 8\sum_{k=1}^{\infty}\left(\frac{1-(-1)^k}{2k}\right)^2 = 8\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right)$

6 If a square pulse is centered at $x = 0$ to give

$$f(x) = 1 \quad \text{for} \quad |x| < \frac{\pi}{2}, \quad f(x) = 0 \quad \text{for} \quad \frac{\pi}{2} < |x| < \pi,$$

draw its graph and find its Fourier coefficients a_k and b_k.

Solution

$$a_0 = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} dx = \frac{1}{2}$$

$$a_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos kx\, dx = \frac{2}{k\pi}\sin\frac{k\pi}{2} = \sin\, c\left(\frac{k\pi}{2}\right)$$

$$b_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin kx\, dx = 0$$

7 Plot the first three partial sums and the function $x(\pi - x)$:

$$x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{27} + \frac{\sin 5x}{125} + \cdots\right), 0 < x < \pi.$$

Why is $1/k^3$ the decay rate for this function? What is its second derivative?

Solution The parabola $y = x(\pi - x) = x\pi - x^2$ starts at $y(0) = 0$ with slope $y'(0) = \pi$ and second derivative $y''(0) = -2$. Its sine series makes it an odd function $x\pi + x^2$ from $-\pi$ to 0. This odd extension has **second derivative** $\boldsymbol{= \pm 2}$. That jump in y'' means that the Fourier coefficients b_k will decay like $1/k^3$. (Remember $1/k$ for jumps in $y(x)$ and $1/k^2$ for jumps in $y'(x)$—no jumps in y, y' for this example.)

8 Sketch the 2π-periodic half wave with $f(x) = \sin x$ for $0 < x < \pi$ and $f(x) = 0$ for $-\pi < x < 0$. Find its Fourier series.

Solution The function is not odd or even, so integrals must go from $-\pi$ to π. The function is zero from $-\pi$ to 0 leaving only these integrals for a_0, a_k, b_k :

$$a_0 = \frac{1}{2\pi}\int_0^\pi \sin x\,dx = \frac{1}{2\pi}\left[-\cos x\right]_0^\pi = \boldsymbol{\frac{1}{\pi}}$$

$$a_k = \frac{1}{\pi}\int_0^\pi \sin x\cos kx\,dx = -\frac{1}{2\pi}\left[\frac{\cos(1-k)x}{1-k} + \frac{\cos(1+k)x}{1+k}\right]_0^\pi =$$

$$[k \text{ even}]\ \frac{1}{\pi}\left(\frac{1}{1-k} + \frac{1}{1+k}\right) = \boldsymbol{\frac{2}{\pi(1-k^2)}} \quad [\text{and } \mathbf{0} \text{ for k odd}]$$

$$b_k = \frac{1}{\pi}\int_0^\pi \sin x\sin kx\,dx \text{ gives } b_1 = \frac{1}{2} \text{ and other } b_k = 0.$$

9 Suppose $G(x)$ has period $2L$ instead of 2π. Then $G(x+2L) = G(x)$. Integrals go from $-L$ to L or from 0 to $2L$. The Fourier formulas change by a factor π/L:

The coefficients in $G(x) = \sum_{-\infty}^{\infty} \boldsymbol{C}_k e^{ik\pi x/L}$ are $\boldsymbol{C}_k = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} G(x)e^{-ik\pi x/L}dx.$

Derive this formula for C_k: Multiply the first equation for $G(x)$ by _____ and integrate both sides. Why is the integral on the right side equal to $2LC_k$?

Solution Multiply $G(x) = \sum_{-\infty}^{\infty} C_k e^{ik\pi x/L}$ by $e^{-ik\pi x/L}$. Integrate.

$$\int_{-L}^{L} G(x)e^{-ik\pi x/L}\,dx = \int_{-L}^{L} e^{-ik\pi x/L}\sum_{-\infty}^{\infty} C_k e^{ik\pi x/L}\,dx$$

$$\int_{-L}^{L} G(x)e^{-ik\pi x/L}\,dx = C_k\int_{-L}^{L} dx = 2LC_k \text{ (orthogonality)}$$

$$C_k = \frac{1}{2L}\int_{-L}^{L} G(x)e^{-ik\pi x/L}\,dx$$

10 For G_{even}, use Problem 9 to find the cosine coefficient A_k from $(C_k + C_{-k})/2$:

$$G_{\text{even}}(x) = \sum_0^\infty A_k \cos\frac{k\pi x}{L} \quad \text{has} \quad A_k = \frac{1}{L}\int_0^L G_{\text{even}}(x)\ \cos\frac{k\pi x}{L}dx.$$

G_{even} is $\frac{1}{2}(G(x) + G(-x))$. Exception for $A_0 = C_0$: Divide by $2L$ instead of L.

Solution The result comes directly from $\frac{1}{2}(C_k + C_{-k})$.

11 Problem 10 tells us that $\boldsymbol{a_k = \frac{1}{2}(c_k + c_{-k})}$ on the usual interval from 0 to π. Find a similar formula for b_k from c_k and c_{-k}. In the reverse direction, find the complex coefficient c_k in $F(x) = \sum c_k e^{ikx}$ from the real coefficients a_k and b_k.

Solution **Solution and correction** We are comparing two ways to write a Fourier series :

$$\sum_{-\infty}^{\infty} c_k e^{ikx} = a_0 + \sum_{1}^{\infty} a_k \cos kx + \sum_{1}^{\infty} b_k \sin kx$$

Pick out the terms for k and $-k$:

$$c_k e^{ikx} + c_{-k} e^{-ikx} = a_k \cos kx + b_k \sin kx$$

Use Euler's formula to reach cosines/sines on both sides :

$$(c_k + c_{-k}) \cos kx + i(c_k - c_{-k}) \sin kx = a_k \cos kx + b_k \sin kx$$

This shows that $\boldsymbol{a_k = c_k + c_{-k}}$ **(correction from text)** and $\boldsymbol{b_k = i(c_k - c_{-k})}$.

Reverse Euler's formula to reach complex exponentials on both sides :

$$c_k e^{ikx} + c_{-k} e^{-ikx} = \frac{1}{2} a_k (e^{ikx} + e^{-ikx}) + \frac{1}{2i} b_k (e^{ikx} - e^{-ikx})$$

This shows that $\boldsymbol{c_k = \frac{1}{2} a_k + \frac{1}{2i} b_k}$ and $\boldsymbol{c_{-k} = \frac{1}{2} a_k - \frac{1}{2i} b_k}$.

Real functions with real a's and b's lead to $c_{-k} = \overline{c_k}$ (complex conjugates)

12 Find the solution to Laplace's equation with $u_0 = \theta$ on the boundary. Why is this the imaginary part of $2(z - z^2/2 + z^3/3 \cdots) = 2 \log(1 + z)$? Confirm that on the unit circle $z = e^{i\theta}$, the imaginary part of $2 \log(1 + z)$ agrees with θ.

Solution The sine series of the odd function $f(\theta) = \theta$ has coefficients $b_n =$

$$\frac{2}{\pi} \int_0^{\pi} \theta \sin n\theta \, d\theta = \frac{2}{\pi} \left[\frac{1}{n^2} \sin n\theta - \frac{\theta}{n} \cos n\theta \right]_0^{\pi} = -\frac{2 \cos n\pi}{n} = \mathbf{2} \left[\frac{\mathbf{1}}{\mathbf{1}}, -\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{3}}, -\frac{\mathbf{1}}{\mathbf{4}}, \cdots \right]$$

The solution to Laplace's equation inside the circle has factors r^n :

$$u(r, \theta) = \sum \boldsymbol{b_n r^n \sin n\theta} = 2r \sin \theta - \frac{2}{2} r^2 \sin 2\theta + \frac{2}{3} r^3 \sin 3\theta \ldots$$
$$= \operatorname{Im} \left[2z - \frac{2}{2} z^2 + \frac{2}{3} z^3 \ldots \right] = \operatorname{Im}[2 \log(1 + z)].$$

13 If the boundary condition for Laplace's equation is $u_0 = 1$ for $0 < \theta < \pi$ and $u_0 = 0$ for $-\pi < \theta < 0$, find the Fourier series solution $u(r, \theta)$ inside the unit circle. What is u at the origin $r = 0$?

Solution This 0-1 step function $u_0(\theta)$ equals $\frac{1}{2} + \frac{1}{2}$ (square wave). Equation (8) of the text gives the Fourier sine series for the square wave :

$$\text{0-1 Step Function } \; u_0(\theta) = \frac{1}{2} + \frac{2}{\pi} \left[\frac{\sin \theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots \right]$$

Then the solution to Laplace's equation includes factors r^n :

$$u(r, \theta) = \frac{1}{2} + \frac{2}{\pi} \left[\frac{r \sin \theta}{1} + \frac{r^3 \sin 3\theta}{3} + \frac{r^5 \sin 5\theta}{5} + \cdots \right] = \frac{\mathbf{1}}{\mathbf{2}} \quad \textbf{at} \quad \boldsymbol{r = 0}.$$

14 With boundary values $u_0(\theta) = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \cdots$, what is the Fourier series solution to Laplace's equation in the circle? Sum this geometric series.

Solution Inside the circle we see factors r^n (and $1 + x + x^2 + \cdots = 1/(1-x)$):

$$u(r,\theta) = 1 + \frac{1}{2}re^{i\theta} + \frac{1}{4}r^2e^{2i\theta} + \cdots = \mathbf{1/}\left(\mathbf{1 - \frac{1}{2}re^{i\theta}}\right).$$

15 (a) Verify that the fraction in Poisson's formula (30) satisfies Laplace's equation.

Solution (a) We could verify Laplace's equation in r, θ coordinates or recognize that every term in the sum (29) solves that equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

(b) Find the response $u(r,\theta)$ to an impulse at $x = 0, y = 1$ (where $\theta = \frac{\pi}{2}$).

Solution (b) When the source is at the point $\theta = \pi$, this replaces $r\cos\theta$ by $-r\cos\theta$ in equation (30). Then the response to a point source is infinite at $r = 1, \theta = \pi$:

$$u(r,\theta) = \mathbf{\frac{1}{2\pi} \quad \frac{1 - r^2}{1 + r^2 + 2r\cos\theta}}$$

16 With complex exponentials in $F(x) = \sum c_k e^{ikx}$, the energy identity (21) changes to $\int_{-\pi}^{\pi} |F(x)|^2\, dx = 2\pi \sum |c_k|^2$. Derive this by integrating $(\sum c_k e^{ikx})(\sum \overline{c}_k e^{-ikx})$.

Solution All products $e^{ikx}e^{-ikx}$ integrate to zero except when $n = k$:

$$\int_{-\pi}^{\pi} (c_k e^{ikx})(\overline{c_k} e^{-ikx})\, dx = 2\pi c_k \overline{c_k} = 2\pi |c_k|^2.$$

The total energy is the sum over all k.

17 A centered square wave has $F(x) = 1$ for $|x| \le \pi/2$.

(a) Find its energy $\int |F(x)|^2\, dx$ by direct integration

Solution (a) $\displaystyle \int |F(x)|^2\, dx = \int_{-\pi/2}^{\pi/2} dx = \boldsymbol{\pi}.$

(b) Compute its Fourier coefficients c_k as specific numbers

Solution (b)

$$c_k = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-ikx}\, dx = \left[\frac{1}{2\pi}\,\frac{e^{-ikx}}{-ik}\right]_{-\pi/2}^{\pi/2}$$
$$= \frac{1}{2\pi\, ik}\left(e^{ik\pi/2} - e^{-ik\pi/2}\right) = \mathbf{\frac{1}{\pi\, k}\sin\left(\frac{k\pi}{2}\right)}$$

(c) Find the sum in the energy identity (Problem 8).

Solution (c) $\sin\dfrac{k\pi}{2} = 1, 0, -1, 0$ (repeated) so $2\pi \sum |c_k|^2 = \dfrac{2}{\pi}\left(\dfrac{1}{1} + \dfrac{1}{9} + \dfrac{1}{25} + \cdots\right) = 1.$

18 $F(x) = 1 + (\cos x)/2 + \cdots + (\cos nx)/2^n + \cdots$ is analytic : infinitely smooth.

(a) If you take 10 derivatives, what is the Fourier series of $d^{10}F/dx^{10}$?

(b) Does that series still converge quickly ? Compare n^{10} with 2^n for $n = 2^{10}$.

Solution (a) 10 derivatives of $\cos\, nx$ gives $-n^{10}\cos\, nx$:

$$\frac{d^{10}F}{dx^{10}} = -\frac{1}{2}\cos x - \frac{2^{10}}{2^2}\cos\, 2x - \frac{3^{10}}{2^3}\cos\, 3x \cdots - \frac{n^{10}}{2^n}\cos\, nx - \cdots$$

Solution (b) Yes, 2^n gets large much faster than n^{10} so the series easily converges.

At $n = 2^{10} = 1024$ we have $2^n = 2^{1024}$, much larger than $n^{10} = 2^{100}$.

19 If $f(x) = 1$ for $|x| \le \pi/2$ and $f(x) = 0$ for $\pi/2 < |x| < \pi$, find its cosine coefficients. Can you graph and compute the Gibbs overshoot at the jumps ?

Solution $a_0 =$ **average value** $= \mathbf{\frac{1}{2}}$

$$a_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos\, kx\, dx = \left[\frac{1}{\pi\, k}\sin\, kx\right]_{-\pi/2}^{\pi/2} = \boldsymbol{\frac{2}{\pi\, k}\sin\frac{k\pi}{2}}$$

20 Find all the coefficients a_k and b_k for F, I, and D on the interval $-\pi \le x \le \pi$:

$$F(x) = \delta\left(x - \frac{\pi}{2}\right) \qquad I(x) = \int_0^x \delta\left(x - \frac{\pi}{2}\right) dx \qquad D(x) = \frac{d}{dx}\delta\left(x - \frac{\pi}{2}\right).$$

Solution (a) Integrate $\cos\, kx$ and $\sin\, kx$ against $\delta(x - \frac{\pi}{2})$ to get

$$a_0 = \boldsymbol{\frac{1}{2\pi}} \qquad a_k = \boldsymbol{\frac{1}{\pi}\cos\frac{k\pi}{2}} \text{ and } b_k = \boldsymbol{\frac{1}{\pi}\sin\frac{k\pi}{2}}$$

Solution (b) The integral $I(x)$ is the unit step function $H(x - \frac{\pi}{2})$ with jump at $x = \frac{\pi}{2}$:

$$a_0 = \frac{1}{2\pi}\int_{\pi/2}^{\pi} 1\, dx = \frac{1}{4}$$

$$a_k = \frac{1}{\pi}\int_{\pi/2}^{\pi} \cos\, kx\, dx = \frac{1}{\pi\, k}\left(\sin\, k\pi - \sin\frac{k\pi}{2}\right) = -\frac{1}{\pi\, k}\sin\frac{k\pi}{2}$$

$$b_k = \frac{1}{\pi}\int_{\pi/2}^{\pi} \sin\, kx\, dx = -\frac{1}{\pi\, k}\left(\cos\, k\pi - \cos\frac{k\pi}{2}\right)$$

Solution (c) $D(x)$ is the "doublet" = derivative of the delta function $\delta\left(x - \frac{\pi}{2}\right)$. You must integrate by parts (and $D(-\pi) = D(\pi) = 0$ fortunately).

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D(x)\cos\, kx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta\left(x - \frac{\pi}{2}\right)(k\sin\, kx)\, dx$$

So a_k for $D(x)$ is $\boldsymbol{kb_k}$ **in part (b)**, and b_k for $D(x)$ is $\boldsymbol{-ka_k}$ **in part (b)**.

21 For the one-sided tall box function in Example 4, with $F = 1/h$ for $0 \le x \le h$, what is its odd part $\frac{1}{2}(F(x) - F(-x))$? I am surprised that the Fourier coefficients of this odd part disappear as h approaches zero and $F(x)$ approaches $\delta(x)$.

Solution Every function has an even part and an odd part:

$$F_{\text{even}}(x) = \frac{1}{2}(F(x) + F(-x)) \quad F_{\text{odd}}(x) = \frac{1}{2}(F(x) - F(-x)) \quad F = F_{\text{even}} + F_{\text{odd}}$$

For the one-sided box function, those even and odd parts are

$$F_{\text{even}}(x) = \frac{1}{2h} \text{ for } |x| \le h \quad F_{\text{odd}}(x) = -\frac{1}{h} \text{ for } -h \le x \le 0, +\frac{1}{h} \text{ for } 0 < x \le h.$$

The Fourier coefficients of F_{odd} don't really "disappear" as $h \to 0$, because the energy $\int |F_{\text{odd}}|^2\,dx$ is growing. But it is growing in the high frequencies and any particular coefficient c_k (at a fixed frequency k) approaches zero as $h \to 0$.

22 Find the series $F(x) = \sum c_k e^{ikx}$ for $F(x) = e^x$ on $-\pi \le x \le \pi$. That function e^x looks smooth, but there must be a hidden jump to get coefficients c_k proportional to $1/k$. Where is the jump?

Solution When e^x is made into a periodic function there is a jump (or a drop) at $x = \pi$. The drop from e^π to $e^{-\pi}$ starts the next 2π-interval. That drop shows up as a factor multiplying the $1/k$ decay that all jump functions show in their Fourier expansion:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ikx}\,dx = \left[\frac{1}{2\pi}\,\frac{e^{(1-ik)x}}{1-ik}\right]_{x=-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\,\frac{\boldsymbol{e^{\pi} - e^{-\pi}}}{1-ik}.$$

23 (a) (Old particular solution) Solve $Ay'' + By' + Cy = e^{ikx}$.

(b) (New particular solution) Solve $Ay'' + By' + Cy = \sum c_k e^{ikx}$.

Solution This problem shows directly the power of **linearity** to deal with complicated forcing functions as combinations of simple forcing functions e^{ikx}:

$$Ay'' + By' + Cy = e^{ikx} \quad \text{has } y_p = \frac{1}{(ik)^2 A + ikB + C}\,e^{ikx} = Y_k e^{ikx}$$

$$Ay'' + By' + Cy = \sum c_k e^{ikx} \quad \text{has } y_p = \sum c_k Y_k e^{ikx}.$$

Problem Set 8.2, page 453

1 Multiply the three matrices in equation (11) and compare with F. In which six entries do you need to know that $i^2 = -1$? This is $(w_4)^2 = w_2$. If $M = N/2$, why is $(w_N)^M = -1$?

Solution

2 *Why is row i of $\overline{F}$ the same as row $N - i$ of F* (numbered from 0 to $N - 1$)?

Solution

3 From Problem 8, find the 4 by 4 permutation matrix P so that $F = P\overline{F}$. Check that $P^2 = I$ so that $P = P^{-1}$. Then from $\overline{F}F = 4I$ show that $F^2 = 4P$.

It is amazing that $F^4 = 16P^2 = 16I$. *Four transforms of any* $\boldsymbol{c}$ *bring back* $\mathbf{16}\,\boldsymbol{c}$*.* For all N, F^2/N is a permutation matrix P and $\boldsymbol{F^4 = N^2 I}$.

Solution

4 Invert the three factors in equation (11) to find a fast factorization of F^{-1}.

5 F is symmetric. Transpose equation (11) to find a new Fast Fourier Transform.

Solution

6 All entries in the factorization of F_6 involve powers of $w =$ sixth root of 1:

$$F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & \\ & F_3 \end{bmatrix} \begin{bmatrix} & P & \end{bmatrix}.$$

Write down these factors with $1, w, w^2$ in D and powers of w^2 in F_3. Multiply!

Solution

7 Put the vector $\boldsymbol{c} = (1, 0, 1, 0)$ through the three steps of the FFT to find $\boldsymbol{y} = F\boldsymbol{c}$. Do the same for $\boldsymbol{c} = (0, 1, 0, 1)$.

Solution

8 Compute $\boldsymbol{y} = F_8\boldsymbol{c}$ by the three FFT steps for $\boldsymbol{c} = (1, 0, 1, 0, 1, 0, 1, 0)$. Repeat the computation for $\boldsymbol{c} = (0, 1, 0, 1, 0, 1, 0, 1)$.

Solution

9 If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the _____ and _____ roots of 1.

Solution

10 F is a symmetric matrix. Its eigenvalues aren't real. How is this possible ?

Solution

The three great symmetric tridiagonal matrices of applied mathematics are K**,** B**,** C. The eigenvectors of K, B, and C are discrete **sines**, **cosines**, and **exponentials**. The eigenvector matrices give the **DST**, **DCT**, and **DFT** — discrete transforms for signal processing. Notice that diagonals of the circulant matrix C loop around to the far corners.

$$\boldsymbol{K} = \begin{bmatrix} \mathbf{2} & -1 & & \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ & & -1 & \mathbf{2} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} \mathbf{1} & -1 & & \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ & & -1 & \mathbf{1} \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 2 & -1 & \cdot & \mathbf{-1} \\ -1 & 2 & -1 & \\ & \cdot & \cdot & \cdot \\ \mathbf{-1} & \cdot & -1 & 2 \end{bmatrix} \qquad \begin{aligned} K_{11} &= K_{NN} = \mathbf{2} \\ B_{11} &= B_{NN} = \mathbf{1} \\ C_{1N} &= C_{N1} = \mathbf{-1} \end{aligned}$$

11 The eigenvectors of K_N and B_N are the discrete sines $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_N$ and the discrete cosines $\boldsymbol{c}_0, \ldots, \boldsymbol{c}_{N-1}$. Notice the eigenvector $\boldsymbol{c}_0 = (1, 1, \ldots, 1)$. Here are $\boldsymbol{s}_k$ and $\boldsymbol{c}_k$—these vectors are samples of $\sin kx$ and $\cos kx$ from 0 to π.

$$\left(\sin\frac{\pi k}{N+1}, \sin\frac{2\pi k}{N+1}, \ldots, \sin\frac{N\pi k}{N+1}\right) \text{ and } \left(\cos\frac{\pi k}{2N}, \cos\frac{3\pi k}{2N}, \ldots, \cos\frac{(2N-1)\pi k}{2N}\right)$$

For 2 by 2 matrices K_2 and B_2, verify that $\boldsymbol{s}_1$, $\boldsymbol{s}_2$ and $\boldsymbol{c}_0$, $\boldsymbol{c}_1$ are eigenvectors.

Solution

12 Show that C_3 has eigenvalues $\lambda = 0, 3, 3$ with eigenvectors $\boldsymbol{e}_0 = (1, 1, 1)$, $\boldsymbol{e}_1 = (1, w, w^2)$, $\boldsymbol{e}_2 = (1, w^2, w^4)$. You may prefer the real eigenvectors $(1, 1, 1)$ and $(1, 0, -1)$ and $(1, -2, 1)$.

Solution

13 Multiply to see the eigenvectors $\boldsymbol{e}_k$ and eigenvalues λ_k of C_N. Simplify to $\lambda_k = 2 - 2\cos(2\pi k/N)$. Explain why C_N is only semidefinite. It is not positive definite.

$$C\boldsymbol{e}_k = \begin{bmatrix} 2 & -1 & & -1 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ -1 & & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ w^k \\ w^{2k} \\ w^{(N-1)k} \end{bmatrix} = (2 - w^k - w^{-k}) \begin{bmatrix} 1 \\ w^k \\ w^{2k} \\ w^{(N-1)k} \end{bmatrix}.$$

Solution

14 The eigenvectors $\boldsymbol{e}_k$ of C are automatically perpendicular because C is a _____ matrix. (To tell the truth, C has repeated eigenvalues as in Problem 12. There was a plane of eigenvectors for $\lambda = 3$ and we chose orthogonal $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in that plane.)

Solution

15 Write the 2 eigenvalues for K_2 and the 3 eigenvalues for B_3. Always K_N and B_{N+1} have the same N eigenvalues, with the extra eigenvalue _____ for B_{N+1}. (This is because $K = A^{\mathrm{T}}A$ and $B = AA^{\mathrm{T}}$.)

Solution

Problem Set 8.5, page 477

1 When the driving function is $f(t) = \delta(t)$, the solution starting from rest is the **impulse response**. The impulse is $\delta(t)$, the response is $y(t)$. Transform this equation to find the **transfer function** $Y(s)$. Invert to find the impulse response $y(t)$.

$$y'' + y = \delta(t) \text{ with } y(0) = 0 \text{ and } y'(0) = 0$$

Solution Take the Laplace Transform of $y'' + y = \delta(t)$ with $y(0) = y'(0) = 0$:

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 1$$

$$Y(s)(s^2 + 1) = 1$$

$$Y(s) = \frac{1}{s^2+1} \text{ is the transform of } y(t) = \mathbf{\sin t}.$$

2 (Important) Find the first derivative and second derivative of $f(t) = \sin t$ for $t \geq 0$. Watch for a jump at $t = 0$ which produces a spike (delta function) in the derivative.

Solution The first derivative of $\sin(t)$ is $\cos(t)$, and the second derivative is $-\sin(t) + \delta(t)$.

3 Find the Laplace transform of the unit box function $b(t) = \{1 \text{ for } 0 \leq t < 1\} = H(t) - H(t-1)$. The unit step function is $H(t)$ in honor of Oliver Heaviside.

Solution The unit box function is $f(t) = H(t) - H(t-1)$

The transform is $F(s) = \dfrac{1}{s} - \dfrac{e^{-s}}{s} = \boldsymbol{\dfrac{1}{s}(1 - e^{-s})}$

The same result comes from $F(s) = \displaystyle\int_0^\infty f(t)\, e^{-st}\, dt = \int_0^1 e^{-st}\, dt.$

4 If the Fourier transform of $f(t)$ is defined by $\widehat{f}(k) = \int f(t) e^{-ikt} dt$ and $f(t) = 0$ for $t < 0$, what is the connection between $\widehat{f}(k)$ and the Laplace transform $F(s)$?

Solution The Fourier Transform is the Laplace Transform with $s = ik$: $\boldsymbol{\widehat{f}(k) = F(ik)}$.

5 What is the Laplace transform $R(s)$ of the standard **ramp function** $r(t) = t$? For $t < 0$ all functions are zero. The derivative of $r(t)$ is the unit step $H(t)$. Then multiplying $R(s)$ by s gives _____.

Solution The Laplace Transform $R(s)$ of the Ramp Function $r(t) = t$ is

$$R(s) = \int_0^\infty te^{-st}\, dt = -\frac{te^{-st}}{s}\Bigg|_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\, dt = 0 - \frac{e^{-st}}{s^2}\Bigg|_0^\infty = \frac{1}{s^2}$$

Multiplying $R(s)$ by s gives the Laplace transform $1/s$ of the step function.

6 Find the Laplace transform $F(s)$ of each $f(t)$, and the poles of $F(s)$:

(a) $f = 1 + t$	(b) $f = t\cos\omega t$	(c) $f = \cos(\omega t - \theta)$
(d) $f = \cos^2 t$	(e) $f = e^{-2t}\cos t$	(f) $f = te^{-t}\sin\omega t$

Solution (a) The transform of $f(t) = 1 + t$ has a **double pole** at $s = 0$:

$$F(s) = \int_0^\infty (1+t)e^{-st}\, dt = \int_0^\infty e^{-st}\, dt + \int_0^\infty te^{-st}\, dt = \frac{1}{s} + \frac{1}{s^2} = \boldsymbol{\frac{1+s}{s^2}}$$

Solution (b)

$$f(t) = t\cos(\omega t) = t\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) = \frac{te^{i\omega t}}{2} + \frac{te^{-i\omega t}}{2} \quad \text{transforms to}$$

$$F(s) = \int_0^\infty \frac{te^{(i\omega - s)t}}{2}\, dt + \int_0^\infty \frac{te^{-(i\omega - s)t}}{2}\, dt$$

$$= \frac{-e^{-t(s-i\omega)}(st - it\omega + 1)}{2(s - i\omega)^2}\Bigg|_0^\infty + \frac{-e^{-t(s+i\omega)}(st + it\omega + 1)}{2(s + i\omega)^2}\Bigg|_0^\infty$$

$$= \frac{1}{2(s - i\omega)^2} + \frac{1}{2(s + i\omega)^2} = \frac{(s - i\omega)^2 + (s + i\omega)^2}{2(s - i\omega)^2(s + i\omega)^2} = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

Poles occur at $s = i\omega$ and $s = -i\omega$, the two exponents of $f(t)$.

Solution (c) $f(t) = \cos(\omega t - \theta) = \cos\omega t\cos\theta + \sin\omega t\sin\theta$ transforms to

$$F(s) = \frac{s}{s^2+\omega^2}\cos\theta + \frac{\omega}{s^2+\omega^2}\sin\theta$$

Poles occur at $s = \pm i\omega$.

Solution (d)

$$f(t) = \cos^2(t) = \frac{1}{4}(e^{it}+e^{-it})^2 = \frac{1}{4}(e^{2it}+2+e^{-2it})$$

$$F(s) = \int_0^\infty \frac{1}{4}(e^{2it}+e^{-2it}+2)e^{-st}\,dt$$

$$= -\frac{1}{4(s-2i)} + \frac{1}{4(s+2i)} + \frac{1}{2s} = \frac{2s}{4(s^2+4)} + \frac{1}{2s} = \frac{s^2+2}{s(s^2+4)}$$

Poles occur at $s = 0$ and $s = \pm 2i$. Another way is to write $\cos^2 t = \dfrac{1+\cos 2t}{2}$

Solution (e)

$$f(t) = e^{-2t}\cos t = \frac{1}{2}e^{(i-2)t} + \frac{1}{2}e^{-(i+2)t}$$

$$F(s) = \int_0^\infty \frac{1}{2}e^{(i-2)t}e^{-st}\,dt + \int_0^\infty \frac{1}{2}e^{-(i+2)t}e^{-st}\,dt$$

$$= \frac{1}{2(-i+2+s)} + \frac{1}{2(i+2+s)} = \frac{s+2}{(s+2)^2+1}$$

Poles occur at the exponents $s = -2 \pm i$ in $f(t)$.

Solution (f)

$$f(t) = te^{-t}\sin\omega t = \frac{t}{2i}e^{(i\omega-1)t} - \frac{t}{2i}e^{-(i\omega+1)t}$$

$$F(s) = \int_0^\infty \left(\frac{t}{2i}e^{(i\omega-1)t} - \frac{t}{2i}e^{-(i\omega+1)t}\right)e^{-st}\,dt$$

$$= \int_0^\infty \frac{t}{2i}e^{(i\omega-1-s)t}\,dt - \int_0^\infty \frac{t}{2i}e^{-(i\omega+1+s)t}\,dt$$

$$= \left.\frac{ie^{-t(s-i\omega+1)}(1+t(s-i\omega+1))}{2(s-i\omega+1)^2} - \frac{ie^{-t(s+i\omega+1)}(1+t(s+i\omega+1))}{2(s+i\omega+1)^2}\right|_0^\infty$$

Poles of $F(s)$ occur at $s = \boldsymbol{-1 \pm i\omega}$, the exponents of $f(t)$.

7 Find the Laplace transform s of $f(t)$ = next integer above t and $f(t) = t\,\delta(t)$.

A staircase $f(t) = [t] = H(t) + H(t-1) + H(t-2) + \cdots$ = next integer above t is a sum of step functions. The transform is

$$\frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \cdots = \frac{1}{s}(1+e^{-s}+e^{-2s}+\cdots) = \frac{1}{s}\left(\frac{1}{1-e^{-s}}\right).$$

The differentiation rule $\mathcal{L}(tf(t)) = -F'(s)$ with $f(t) = \delta(t)$ and $F(s) = 1$ gives

$$\mathcal{L}(t\delta(t)) = -\frac{d}{ds}(1) = \mathbf{0}$$ (this is correct because $t\delta(t)$ is the zero function).

8 *Inverse Laplace Transform*: Find the function $f(t)$ from its transform $F(s)$:

(a) $\dfrac{1}{s-2\pi i}$ (b) $\dfrac{s+1}{s^2+1}$ (c) $\dfrac{1}{(s-1)(s-2)}$

(d) $1/(s^2+2s+10)$ (e) $e^{-s}/(s-a)$ (f) $2s$

Solution (a) $F(s) = \dfrac{1}{s-2\pi i}$ is the transform of $f(t) = \boldsymbol{e^{2\pi i t}}$.

Solution (b) $F(s) = \dfrac{s}{s^2+1} + \dfrac{1}{s^2+1}$ is the transform of $f(t) = \mathbf{\cos t + \sin t}$.

Solution (c) $F(s) = \dfrac{1}{(s-1)(s-2)} = \dfrac{1}{s-2} - \dfrac{1}{s-1}$ is the transform of $f(t) = \boldsymbol{e^{2t} - e^{t}}$.

Solution (d)

$$\begin{aligned} F(s) &= \frac{1}{s^2+2s+10} = \frac{1}{(s+1+3i)(s+1-3i)} \\ &= \frac{i}{6(s+(1+3i))} - \frac{i}{6(s+(1-3i))} \\ f(t) &= \frac{i}{6}e^{-(1+3i)t} - \frac{i}{6}e^{-(1-3i)t} \\ &= -\boldsymbol{\frac{e^{-t}\sin(3t)}{3}} \end{aligned}$$

Solution (e)

$$\begin{aligned} F(s) &= \frac{e^{-s}}{s-a} \\ f(t) &= e^{a(t-1)}H(t-1) = \quad \textbf{shift of } \boldsymbol{e^{at}} \end{aligned}$$

Solution (f)

$$\begin{aligned} F(s) &= 2s \\ f(t) &= \boldsymbol{2\, d\delta/dt} \end{aligned}$$

9 Solve $y''+y=0$ from $y(0)$ and $y'(0)$ by expressing $Y(s)$ as a combination of $s/(s^2+1)$ and $1/(s^2+1)$. Find the inverse transform $y(t)$ from the table.

Solution

$$\begin{aligned} y'' + y &= 0 \\ s^2Y(s) - sy(0) - y'(0) + Y(s) &= 0 \\ Y(s)(s^2+1) &= sy(0) + y'(0) \\ Y(s) &= y(0)\frac{s}{s^2+1} + y'(0)\frac{1}{s^2+1} \end{aligned}$$

The inverse transform is $y(t) = \boldsymbol{y(0)\cos(t) + y'(0)\sin(t)}$.

10 Solve $y''+3y'+2y=\delta$ starting from $y(0)=0$ and $y'(0)=1$ by Laplace transform. Find the poles and partial fractions for $Y(s)$ and invert to find $y(t)$.

Solution The transform of $\dfrac{d^2y}{dt^2} + 3\,\dfrac{dy}{dt} + 2y = \delta(t)$ with $y(0)=0$ and $y'(0)=1$ is

$$
\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) &= 1 \\
Y(s)(s^2+3s+2) - 1 &= 1 \\
Y(s) &= \frac{2}{(s+1)(s+2)} \\
Y(s) &= \frac{2}{s+1} - \frac{2}{s+2} \\
y(t) &= \mathbf{2e^{-t} - 2e^{-2t}}
\end{aligned}
$$

11 Solve these initial-value problems by Laplace transform :

(a) $y' + y = e^{i\omega t}, y(0) = 8$ (b) $y'' - y = e^t,\ y(0) = 0,\ y'(0) = 0$

(c) $y' + y = e^{-t}, y(0) = 2$ (d) $y'' + y = 6t,\ y(0) = 0,\ y'(0) = 0$

(e) $y' - i\omega y = \delta(t), y(0) = 0$ (f) $my'' + cy' + ky = 0,\ y(0) = 1, y'(0) = 0$

Solution (a)

$$
\begin{aligned}
y' + y &= e^{i\omega t} \text{ with } y(0) = 8 \\
sY(s) - 8 + Y(s) &= \frac{1}{s - i\omega} \\
Y(s)(s+1) &= \frac{1}{s - i\omega} + 8 \\
Y(s) &= \frac{1}{(s+1)(s-i\omega)} + \frac{8}{s+1} \\
Y(s) &= \frac{1}{1+i\omega}\left(\frac{1}{s-i\omega} - \frac{1}{s+1}\right) + \frac{8}{s+1} \\
\text{Particular} \quad + \quad \text{null} \ \ y(t) &= \boldsymbol{\frac{1}{1+i\omega}\left(e^{i\omega t} - e^{-t}\right) + 8e^{-t}}
\end{aligned}
$$

Solution (b)

$$
\begin{aligned}
y'' - y &= e^t \text{ with } y(0) = 0 \text{ and } y'(0) = 0 \\
s^2Y(s) - Y(s) &= \frac{1}{s-1} \\
Y(s) &= \frac{1}{(s-1)(s+1)(s-1)} \\
&= \frac{1}{4(s+1)} - \frac{1}{4(s-1)} + \frac{1}{2(s-1)^2} \\
y(t) &= \boldsymbol{\frac{e^{-t}}{4} - \frac{e^t}{4} + \frac{te^t}{2}}
\end{aligned}
$$

Solution (c)

$$
\begin{aligned}
y' + y &= e^{-t} \text{ with } y(0) = 2 \\
sY(s) - 2 + Y(s) &= \frac{1}{s+1} \\
Y(s) &= \frac{1}{(s+1)^2} + \frac{2}{s+1} \\
y(t) &= \boldsymbol{te^{-t} + 2e^{-t}}
\end{aligned}
$$

Solution (d)

$$\begin{aligned} y'' + y &= 6t \text{ with } y(0) = y'(0) = 0 \\ s^2Y(s) + Y(s) &= \frac{6}{s^2} \\ Y(s)(s^2+1) &= \frac{6}{s^2} \\ Y(s) &= \frac{6}{s^2} - \frac{3i}{s+i} + \frac{3i}{s-i} \\ y(t) &= 6t - 3ie^{-it} + 3ie^{it} = \mathbf{6t - 6\sin t} \end{aligned}$$

Solution (e)

$$\begin{aligned} y' - i\omega y &= \delta(t) \text{ with } y(0) = 0 \\ sY(s) - i\omega Y(s) &= 1 \\ Y(s) &= \frac{1}{s - i\omega} \\ y(t) &= \boldsymbol{e^{i\omega t}} \end{aligned}$$

Solution (f) $my'' + cy' + ky = 0$ with $y(0) = 1$ and $y'(0) = 0$

$ms^2Y(s) - msy(0) + csY(s) - cy(0) + kY(s) = 0$

$Y(s)(ms^2 + cs + k) = ms + c$

$Y(s) = \dfrac{ms + c}{ms^2 + cs + k}$ has the form $\dfrac{a}{s - s_1} + \dfrac{b}{s - s_2}$

We used this *Mathematica* command to find $y(t)$

Simplify[InverseLaplaceTransform$[(m * s\,+\,c)/(m * s\hat{}2\,+\,c * s\,+\,k),\ s,\ t]]$

$$y(t) = \frac{e^{-\frac{\left(c+\sqrt{c^2-4km}\right)t}{2m}}\left(c\left(-1+e^{\frac{\sqrt{c^2-4kmt}}{m}}\right)+\left(1+e^{\frac{\sqrt{c^2-4kmt}}{m}}\right)\sqrt{c^2-4km}\right)}{2\sqrt{c^2-4km}}$$

12 The transform of e^{At} is $(sI - A)^{-1}$. Compute that matrix (the transfer function) when $A = [1\ \ 1;\ 1\ \ 1]$. Compare the poles of the transform to the eigenvalues of A.

Solution When $A = [1\ \ 1; 1\ \ 1]$ we have:

$$(sI - A)^{-1} = \begin{bmatrix} s-1 & -1 \\ -1 & s-1 \end{bmatrix}^{-1} = \frac{\mathbf{1}}{\boldsymbol{s^2 - 2s}} \begin{bmatrix} \boldsymbol{s-1} & \mathbf{1} \\ \mathbf{1} & \boldsymbol{s-1} \end{bmatrix}.$$

The poles of the system are $s = 2$ and $s = 0$, the eigenvalues of A.

13 If dy/dt decays exponentially, show that $sY(s) \to y(0)$ as $s \to \infty$.

Solution

$$\begin{aligned} sY(s) &= \int_0^\infty se^{-st}y(t)\,dt \quad \textit{(integrate by parts)} \\ &= \int_0^\infty e^{-st}\frac{dy}{dt}\,dt - \left[e^{-st}y(t)\right]_0^\infty \\ &= \int_0^\infty e^{-st}\frac{dy}{dt}\,dt + y(0) \to y(0) \text{ as } s \to \infty \end{aligned}$$

$$\text{Example: } \frac{dy}{dt} = e^{-at} \text{ has } sY(s) - y(0) = \frac{1}{s+a} \to 0 \text{ as } s \to \infty$$

14 Transform Bessel's time-varying equation $ty''+y'+ty=0$ using $\mathscr{L}[ty]=-dY/ds$ to find a first-order equation for Y. By separating variables or by substituting $Y(s)=C/\sqrt{1+s^2}$, find the Laplace transform of the Bessel function $y=J_0$.

Solution The transform of ty'' applies the $\mathscr{L}(t,y)$ rule to y'' instead of y:

$$\mathscr{L}(t,y'')=-\frac{d}{ds}(\text{transform of}y'')\ =-\frac{d}{ds}(s^2Y(s)-sy(0)-y'(0)).$$

Apply this to the transform of $t\dfrac{d^2y}{dt^2}+\dfrac{dy}{dt}+ty=0$

$$-2sY(s)-s^2\frac{dY}{ds}+y(0)+sY(s)-y(0)-\frac{dY}{ds}=0$$

$$-sY(s)-s^2\frac{dY}{ds}-\frac{dY}{ds}=0$$

$$sY(s)=-(s^2+1)\frac{dY}{ds}$$

$$\frac{dY}{Y(s)}=-\frac{s\,ds}{s^2+1}$$

$$\log Y(s)=\log\left(\frac{1}{\sqrt{s^2+1}}\right)$$

The transform of the Bessel solution $y=J_0$ is Y(s)=$\boldsymbol{\dfrac{1}{\sqrt{s^2+1}}}$

15 Find the Laplace transform of a single arch of $f(t)=\sin\pi t$.

Solution A single arch of $\sin\pi t$ extends from $t=0$ to $t=1$:

$$F(s)=\int_0^\infty f(t)e^{-st}dt=\int_0^1\sin(\pi t)e^{-st}dt=\int_0^1\frac{e^{i\pi t-st}}{2i}dt-\int_0^1\frac{e^{-i\pi t-st}}{2i}dt$$

$$=\left[\frac{e^{i\pi t-st}}{2i(i\pi-s)}+\frac{e^{-i\pi t-st}}{2i(i\pi+s)}\right]_{t=0}^{t=1}$$

$$=\frac{e^{i\pi-s}-1}{2i(i\pi-s)}+\frac{e^{-i\pi-s}-1}{2i(i\pi+s)}$$

$$=\left(\frac{-e^{-s}-1}{2i}\right)\left(\frac{1}{i\pi-s}-\frac{1}{i\pi+s}\right)=\left(\frac{e^{-s}+1}{i}\right)\left(\frac{s}{\pi^2+s^2}\right)$$

A faster and more direct approach: One arch of the sine curve agrees with $\sin\pi t+$ unit shift of $\sin\pi t$, because those cancel after one arch.

$$\sin\pi t+\sin\pi(t-1)=\sin\pi t+\sin\pi t\cos\pi=\sin\pi t-\sin\pi t=0.$$

16 Your acceleration $v'=c(v^*-v)$ depends on the velocity v^* of the car ahead:

(a) Find the ratio of Laplace transforms $V^*(s)/V(s)$.

(b) If that car has $v^*=t$ find your velocity $v(t)$ starting from $v(0)=0$.

Solution (a) Take the Laplace Transform of $\dfrac{dv}{dt}=c(v^*-v)$ assuming $v(0)=0$;

$$\begin{aligned} sV(s) - v(0) &= cV^*(s) - cV(s) \\ V(s)(s+c) &= cV^*(s) \\ \frac{V^*(s)}{V(s)} &= \boldsymbol{\frac{s+c}{c}} \end{aligned}$$

Solution (b) If $v^*(t) = t$ then $V^*(s) = \dfrac{1}{s^2}$. Therefore

$$\begin{aligned} V(s)(s+c) &= \frac{c}{s^2} \\ V(s) &= \frac{c}{s^3 + cs^2} \\ &= \frac{1}{c(s+c)} - \frac{1}{cs} + \frac{1}{s^2} \\ v(t) &= \boldsymbol{\frac{e^{-ct}}{c} - \frac{1}{c} + t} \end{aligned}$$

17 A line of cars has $v_n' = c[v_{n-1}(t-T) - v_n(t-T)]$ with $v_0(t) = \cos \omega t$ in front.

(a) Find the growth factor $A = 1/(1 + i\omega e^{i\omega T}/c)$ in oscillation $v_n = A^n e^{i\omega t}$.

(b) Show that $|A| < 1$ and the amplitudes are safely decreasing if $cT < \frac{1}{2}$.

(c) If $cT > \frac{1}{2}$ show that $|A| > 1$ (dangerous) for small ω. (Use $\sin\theta < \theta$.)

Human reaction time is $T \geq 1\,\mathrm{sec}$ and human aggressiveness is $c = 0.4/\mathrm{sec}$.

Danger is pretty close. Probably drivers adjust to be barely safe.

Solution (a) $\dfrac{dv_n}{dt} = c(v_{n-1}(t-T) - v_n(t-T))$ with $v_n = A^n e^{i\omega t}$

$$\begin{aligned} i\omega A^n e^{i\omega t} &= cA^{n-1} e^{i\omega(t-T)} - cA^n e^{i\omega(t-T)} \\ A\frac{i\omega e^{i\omega T}}{c} &= 1 - A \\ A\left(1 + \frac{i\omega e^{i\omega T}}{c}\right) &= 1 \end{aligned}$$

Solution (b)

$$\text{For } |A| < 1 \text{ we need } \left|1 + \frac{i\omega}{c} e^{i\omega T}\right| > 1$$

$$\left|1 - \frac{\omega}{c}\sin(\omega T) + \frac{\omega}{c}\cos(\omega T)\right| > 1$$

$$\left(1 - \frac{\omega}{c}\sin(\omega T)\right)^2 + \frac{\omega^2}{c^2}\cos^2(\omega T) > 1$$

$$1 - \frac{2\omega}{c}\sin(\omega T) + \frac{\omega^2}{c^2}\sin^2(\omega T) + \frac{\omega^2}{c^2}\cos^2(\omega T) > 1$$

$$1 - \frac{2\omega}{c}\sin(\omega T) + \frac{\omega^2}{c^2} > 1$$

$$\frac{\omega^2}{c^2} > \frac{2\omega}{c}\sin(\omega T)$$

$$\text{Since } \sin\omega T < \omega T, \text{ we are safe if } \frac{\omega^2}{c^2} > \frac{2\omega}{c}\omega T \text{ which is } cT < \frac{1}{2}$$

Solution (c) $\sin\omega T \approx \omega T$ when this number is small. Then the same steps show $|A| > 1$ when $cT > \frac{1}{2}$.

18 For $f(t) = \delta(t)$, the transform $F(s) = 1$ is the limit of transforms of tall thin box functions $b(t)$. The boxes have width $\epsilon \to 0$ and height $1/\epsilon$ and area 1.

$$\text{Inside integrals, } b(t) = \left\{ \begin{array}{ll} 1/\epsilon & \text{for } 0 \le t < \epsilon \\ 0 & \text{otherwise} \end{array} \right\} \text{approaches } \delta(t).$$

Find the transform $B(s)$, depending on ϵ. Compute the limit of $B(s)$ as $\epsilon \to 0$.

Solution We begin by finding the transform of the box:

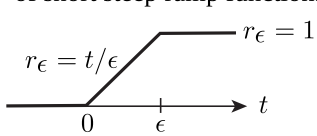
$$B(s) = \int_0^\epsilon \frac{1}{\epsilon} e^{-st}\, dt = \left. \frac{-1}{s\epsilon} e^{-st} \right|_0^\epsilon = \frac{1 - e^{-s\epsilon}}{s\epsilon}$$

We take the limit as $\epsilon \to 0$—the box approaches a delta function!

$$B_\epsilon(s) = \lim_{\epsilon \to 0} \frac{1 - e^{-s\epsilon}}{s\epsilon}$$

$$= \lim \frac{1 - (1 - s\epsilon + \frac{1}{2}s^2\epsilon^2 - \cdots)}{s\epsilon} = \mathbf{1}.$$

19 The transform $1/s$ of the unit step function $H(t)$ comes from the limit of the transforms of short steep ramp functions $r_\epsilon(t)$. These ramps have slope $1/\epsilon$:



Compute $R_\epsilon(s) = \int_0^\epsilon \frac{t}{\epsilon}\, e^{-st} dt + \int_\epsilon^\infty e^{-st} dt$. Let $\epsilon \to 0$.

Solution $$R_\epsilon(s) = \int_0^\epsilon \frac{t}{\epsilon} e^{-st}\, dt + \int_\epsilon^\infty e^{-st}\, dt = \left[\frac{e^{-st}(-st - 1)}{\epsilon s^2}\right]_{t=0}^{t=\epsilon} + \left[\frac{e^{-st}}{-s}\right]_{t=\epsilon}^{t=\infty}$$

$$= \frac{e^{-s\epsilon}(-s\epsilon - 1) + 1}{\epsilon s^2} + \frac{e^{-s\epsilon}}{s} = \boldsymbol{\frac{1 - e^{-s\epsilon}}{\epsilon s^2}}$$

$$\lim R_\epsilon(s) = \lim \frac{1 - (1 - s\epsilon + \frac{1}{2}s^2\epsilon^2 - \cdots)}{\epsilon s^2} = \boldsymbol{\frac{1}{s}}.$$

20 In Problems 18 and 19, show that the derivative of the ramp function $r_\epsilon(t)$ is the box function $b(t)$. The "generalized derivative" of a step is the _____ function.

Solution The generalized derivative of the short ramp $r_\epsilon(t)$ is the thin box $b(t)/\epsilon$. We say "generalized" because this is not a true derivative at $t = 0$: the ramp has zero slope left of $t = 0$ and nonzero slope right of $t = 0$. But the transforms of r_ϵ and b_ϵ follow the rule for derivatives.

The generalized derivative of a step function is a **delta** function.

21 What is the Laplace transform of $y'''(t)$ when you are given $Y(s)$ and $y(0), y'(0), y''(0)$?

Solution The Laplace Transform of $y'''(t)$ is $s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$

22 The *Pontryagin maximum principle* says that the optimal control is "bang-bang"— it only takes on the extreme values permitted by the constraints. To go from rest at $x = 0$ to rest at $x = 1$ in minimum time, use maximum acceleration A and deceleration $-B$. At what time t do you change from the accelerator to the brake ? (This is the fastest driving between two red lights.)

Solution The maximum principle requires full acceleration A to an unknown time t_0 and then full deceleration $-B$ to reach $x = 1$ with zero velocity. The velocities are

$$v = At \quad \text{for} \quad t \le t_0$$

$$v = At_0 - B(t - t_0) \quad \text{for} \quad t > t_0$$

Integrating the velocity $v = dx/dt$ gives the distance $x(t)$:

$$x = \tfrac{1}{2}At^2 \quad \text{for} \quad t < t_0$$

$$x = \tfrac{1}{2}At_0^2 \quad \text{at} \quad t = t_0$$

$$x = \tfrac{1}{2}At_0^2 + At_0(t - t_0) - \tfrac{1}{2}B(t - t_0)^2 \quad \text{for} \quad t > t_0$$

At the final time T we reach $x = 1$ with velocity $v = 0$. This gives two equations for t_0 and T :

$$v = At_0 - B(T - t_0) = 0$$

$$x = At_0T - \tfrac{1}{2}At_0^2 - \tfrac{1}{2}B(T - t_0)^2 = 1$$

Substitute $T = \frac{1}{B}t_0(A + B)$ from the first equation into the second equation. This leaves an ordinary quadratic equation to solve for t_0.

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1 Find the convolution $\boldsymbol{v} * \boldsymbol{w}$ and also the cyclic convolution $\boldsymbol{v} \circledast \boldsymbol{w}$:

(a) $\boldsymbol{v} = (1, 2)$ and $\boldsymbol{w} = (2, 1)$

Solution (a)

Convolution: $(1, 2) * (2, 1)$ $$\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

Cyclic Convolution: $$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

(b) $\boldsymbol{v} = (1, 2, 3)$ and $\boldsymbol{w} = (4, 5, 6)$.

Solution (b)

$$(1,2,3) * (4,5,6) \qquad \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \\ 28 \\ 27 \\ 18 \end{bmatrix}$$

$$\text{Cyclic Convolution:} \qquad \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 31 \\ 31 \\ 28 \end{bmatrix}$$

2 Compute the convolution $(1, 3, 1) * (2, 2, 3) = (a, b, c, d, e)$. To check your answer, add $a + b + c + d + e$. That total should be 35 since $1 + 3 + 1 = 5$ and $2 + 2 + 3 = 7$ and $5 \times 7 = 35$.

Solution

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 11 \\ 11 \\ 3 \end{bmatrix}$$

$1 + 3 + 1$ times $2 + 2 + 3$ is $2 + 8 + 11 + 11 + 3\ :\ (5)(7) = (35)$.

3 Multiply $1 + 3x + x^2$ times $2 + 2x + 3x^2$ to find $a + bx + cx^2 + dx^3 + ex^4$. Your multiplication was the same as the convolution $(1, 3, 1) * (2, 2, 3)$ in Problem 8. When $x = 1$, your multiplication shows why $1 + 3 + 1 = 5$ times $2 + 2 + 3 = 7$ agrees with $a + b + c + d + e = 35$.

Solution

$$(1 + 3x + x^2) \times (2 + 2x + 3x^2) = 2 + 2x + 3x^2 + 6x + 6x^2 + 9x^3 + 2x^2 + 2x^3 + 3x^4$$
$$= \boldsymbol{2 + 8x + 11x^2 + 11x^3 + 3x^4}$$

At $x = 1$ this is again $(5) \times (7) = (35)$.

4 (Deconvolution) Which vector $\boldsymbol{v}$ would you convolve with $\boldsymbol{w} = (1, 2, 3)$ to get $\boldsymbol{v} * \boldsymbol{w} = (0, 1, 2, 3, 0)$? Which $\boldsymbol{v}$ gives $\boldsymbol{v} \circledast \boldsymbol{w} = (3, 1, 2)$?

Solution

$$\begin{bmatrix} v_0 & 0 & 0 \\ v_1 & v_0 & 0 \\ v_2 & v_1 & v_0 \\ 0 & v_2 & v_1 \\ 0 & 0 & v_2 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

The first and last equation give $v_0 = v_2 = 0$. Substituting into the second, third, fourth equation gives $v_1 = 1$. Therefore $v = (\mathbf{0, 1, 0})$.

$$\text{For cyclic convolution} \quad \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_0 & v_2 & v_1 \\ v_1 & v_0 & v_2 \\ v_2 & v_1 & v_0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{gives} \quad \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

5 (a) For the periodic functions $f(x) = 4$ and $g(x) = 2\cos x$, show that $f * g$ is **zero** (the zero function)!

Solution (a) From equation (4) we have

$$(f * g)(x) = \int_0^{2\pi} g(y) f(x-y)\, dy = 4 \int_0^{2\pi} 2\cos y\, dy = 4 \cdot 0 = 0 \text{ for all } x.$$

(b) In frequency space (k-space) you are multiplying the Fourier coefficients of 4 and $2\cos x$. Those coefficients are $c_0 = 4$ and $d_1 = d_{-1} = 1$. Therefore every product $c_k d_k$ is _____.

Solution (b) In frequency space (k-space) you are multiplying the Fourier coefficients of 4 and $2\cos x$. Those coefficients are $c_0 = 4$ and $d_1 = d_{-1} = 1$. **Therefore every product $\boldsymbol{c_k d_k}$ is zero**. These are the coefficients of the zero function.

6 For periodic functions $f = \sum c_k e^{ikx}$ and $g = \sum d_k e^{ikx}$, the Fourier coefficients of $f * g$ are $\boldsymbol{2\pi c_k d_k}$. Test this factor 2π when $f(x) = 1$ and $g(x) = 1$ by computing $f * g$ from its definition (6.4).

Solution From equation (4):

$$(f * g)(x) = \int_0^{2\pi} f(y) g(x-y)\, dy = \int_0^{2\pi} 1 \cdot 1\, dy = 2\pi.$$

The same convolution in k-space has $c_0 = 1$ and $d_0 = 1$ (all other $c_k = d_k = 0$). Then $2\pi c_k d_k$ gives the correct coefficients (2π and 0) of the convolution $f * g$ (which equals 2π).

7 Show by integration that the periodic convolution $\int_0^{2\pi} \cos x \cos(t-x) dx$ is $\pi \cos t$. In k-space you are squaring Fourier coefficients $c_1 = c_{-1} = \frac{1}{2}$ to get $\frac{1}{4}$ and $\frac{1}{4}$; these are the coefficients of $\frac{1}{2}\cos t$. The 2π in Problem 8 makes $\pi \cos t$ correct.

Solution

$$\int_0^{2\pi} \cos x \cos(t-x)\, dx = \int_0^{2\pi} \cos x(\cos t \cos x + \sin t \sin x)\, dx = \pi \cos t + 0.$$

8 Explain why $f * g$ is the same as $g * f$ (periodic or infinite convolution).

Solution In Fourier space convolution $f * g$ or $f \circledast g$ leads to multiplication $c_k d_k$, which is certainly the same as $d_k c_k$. So $f \circledast g = g \circledast f$ in x-space.

9 What 3 by 3 circulant matrix C produces cyclic convolution with the vector $\boldsymbol{c} = (1, 2, 3)$? Then $C\boldsymbol{d}$ equals $\boldsymbol{c} \circledast \boldsymbol{d}$ for every vector $\boldsymbol{d}$. Compute $\boldsymbol{c} \circledast \boldsymbol{d}$ for $\boldsymbol{d} = (0, 1, 0)$.

Solution The circulant matrix $C = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ gives cyclic convolution with $(1, 2, 3)$.

When $d = (0, 1, 0)$ we have $\boldsymbol{c} \circledast \boldsymbol{d} = C\boldsymbol{d} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

10 What 2 by 2 circulant matrix C produces cyclic convolution with $\boldsymbol{c} = (1,1)$? Show in four ways that this C is not invertible. Deconvolution is impossible.

(1) Find the determinant of C. (2) Find the eigenvalues of C.

(3) Find $\boldsymbol{d}$ so that $C\boldsymbol{d} = \boldsymbol{c} \circledast \boldsymbol{d}$ is zero. (4) $F\boldsymbol{c}$ has a zero component.

Solution The 2 by 2 circulant matrix $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ gives $(1,1) \circledast \boldsymbol{d} = C\boldsymbol{d}$.

(1) The determinant of this matrix is zero.

(2) The eigenvalues of C come from $\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1 = 0.$
Then $(1-\lambda)^2 = 1$ and $\lambda = 0, 2$. That zero eigenvalue means that the matrix C is singular.

(3) $C\boldsymbol{d} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so C is not invertible: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ in nullspace.

(4) The Fourier matrix F gives $F\boldsymbol{c} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. This again shows $\lambda = 2$ and 0.

11 (a) Change $b(x) * \delta(x-1)$ to a multiplication $\widehat{b}(k)\ \widehat{d}(k)$:

The box $b(x) = \{1 \text{ for } 0 \le x \le 1\}$ transforms to $\widehat{b}(k) = \int_0^1 e^{-ikx} dx$.

The shifted delta transforms to $\widehat{d}(k) = \int \delta(x-1) e^{-ikx} dx$.

(b) Show that your result $\widehat{b}\ \widehat{d}$ is the transform of a shifted box function. This shows how convolution with $\delta(x-1)$ shifts the box.

Solution This question shows that continuous convolution with $\delta(x-1)$ produces a shift in the box function $b(x)$, just like discrete convolution with the shifted delta vector $(\ldots, 0, 0, 1, \ldots)$ produces a one-step shift.

We compute $\delta(x-1) * b(x)$ in x-space to find $b(x-1)$, or in k-space to see the effect on the coefficients:

$$\widehat{b}(k) = \int_0^1 e^{-ikx} dx = \left[\frac{e^{-ikx}}{-ik} \right]_{x=0}^{x=1} = \frac{1 - e^{-ik}}{ik}$$

$$\textbf{Shifted box}\quad e^{-ik} \left(\frac{1 - e^{-ik}}{ik} \right) \quad \text{agrees with} \quad \int_1^2 e^{-ikx} dx = \left[\frac{e^{-ikx}}{-ik} \right]_{x=1}^{x=2}.$$

12 Take the Laplace transform of these equations to find the transfer function $G(s)$:

(a) $Ay'' + By' + Cy = \delta(t)$ (b) $y' - 5y = \delta(t)$ (c) $2y(t) - y(t-1) = \delta(t)$

Solution (a) $As^2 Y(s) + BsY(s) + CY(s) = 1$ gives the transfer function $\dfrac{1}{As^2 + Bs + C}$

Solution (b) $sY(s) - 5Y(s) = 1$ gives the transfer function $Y(s) = \dfrac{1}{s-5}$

Solution (c) $2Y(s) - Y(s)e^{-s} = 1$ gives the transfer function $Y(s) = \dfrac{1}{2 - e^{-s}}$

13 Take the Laplace transform of $y'''' = \delta(t)$ to find $Y(s)$. From the Transform Table in Section 8.5 find $y(t)$. You will see $y''' = 1$ and $y'''' = 0$. But $y(t) = 0$ for negative t, so your y''' is actually a unit step function and your y'''' is actually $\delta(t)$.

Solution $y'''' = \delta$ transforms to $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = 1$

Assume zero initial values to get $s^4Y(s) = 1$ and $Y(s) = \dfrac{1}{s^4}$ and $y^3 = \dfrac{t^3}{6}$.

This is also the solution to $y'''' = 0$ with initial values $y, y', y'', y''' = \mathbf{0, 0, 0, 1}$.

14 Solve these equations by Laplace transform to find $Y(s)$. Invert that transform with the Table in Section 8.5 to recognize $y(t)$.

(a) $y' - 6y = e^{-t}$, $y(0) = 2$ (b) $y'' + 9y = 1$, $y(0) = y'(0) = 0$.

Solution (a) The transform of $y' - 6y = e^{-t}$ with $y(0) = 2$ is

$$\begin{aligned} sY(s) - 2 - 6Y(s) &= \frac{1}{s+1} \\ Y(s) &= \frac{2}{s-6} + \frac{1}{(s+1)(s-6)} \\ &= \frac{2}{s-6} + \frac{1}{7(s-6)} - \frac{1}{7(s+1)} \\ &= \frac{15}{7(s-6)} - \frac{1}{7(s+1)} \end{aligned}$$

The inverse transform is $y(t) = \boldsymbol{\dfrac{15}{7}e^{6t} - \dfrac{1}{7}e^{-t}}$

Solution (b) The transform of $y'' + 9y = 1$ with $y(0) = y'(0) = 0$ is

$$\begin{aligned} s^2Y(s) + 9Y(s) &= \frac{1}{s} \\ Y(s) &= \frac{1}{s(s^2+9)} \\ &= \frac{1}{9s} - \frac{1}{18(-3i+s)} - \frac{1}{18(3i+s)} \end{aligned}$$

The inverse transform is $y(t) = \dfrac{1}{9} - \dfrac{1}{18}e^{3it} - \dfrac{1}{18}e^{-3it} = \boldsymbol{y_p + y_n}$.

15 Find the Laplace transform of the shifted step $H(t-3)$ that jumps from 0 to 1 at $t = 3$. Solve $y' - ay = H(t-3)$ with $y(0) = 0$ by finding the Laplace transform $Y(s)$ and then its inverse transform $y(t)$: one part for $t < 3$, second part for $t \geq 3$.

Solution The transform of $H(t-3)$ multiplies e^{-3s} by the transform $\frac{1}{s}$ of $H(t)$.

$$y' - ay = H(t-3) \qquad y(0) = 0$$

$$\begin{aligned} sY(s) - aY(s) &= \frac{e^{-3s}}{s} \\ Y(s) &= \frac{e^{-3s}}{s(s-3)} = \frac{e^{-3x}}{3}\left(\frac{1}{s-3} - \frac{1}{s}\right). \end{aligned}$$

The inverse transform $y(t)$ is the shift of $\frac{1}{3}\left(e^{-3t} - 1\right)$: zero until $t = 3$.

16 Solve $y' = 1$ with $y(0) = 4$—a trivial question. Then solve this problem the slow way by finding $Y(s)$ and inverting that transform.

Solution The trivial solution is : $y = t + 4$. The transform method gives

$$\begin{aligned} sY(s) - 4 &= \frac{1}{s} \\ Y(s) &= \frac{1}{s^2} + \frac{4}{s} \\ y(t) &= t + 4 \end{aligned}$$

17 The solution $y(t)$ is the convolution of the input $f(t)$ with what function $g(t)$?

(a) $y' - ay = f(t)$ with $y(0) = 3$

Solution (a)

$$\begin{aligned} y' - ay &= f(t) \qquad \text{with } y(0) = 3 \\ sY(s) - 3 - aY(s) &= F(s) \\ Y(s) &= \frac{3 + F(s)}{s - a} \\ y(t) &= \boldsymbol{3e^{-t} + f(t) * e^{-at}} \end{aligned}$$

(b) $y' -$ (integral of y) $= f(t)$.

Solution (b) The transform of $y'-$ (integral of y) $= f(t)$ is $sY(s) - \dfrac{Y(s)}{s} = F(s)$, if $y(0) = 0$.

The inverse transform of $\dfrac{1}{s - \frac{1}{s}} = \dfrac{s}{s^2 - 1}$ is $\cos(it)$.

Then $Y(s) = \dfrac{F(s)}{s - \frac{1}{s}}$ is the transform of the convolution $f(t) * \cos(it)$.

18 For $y' - ay = f(t)$ with $y(0) = 3$, we could replace that initial value by adding $3\delta(t)$ to the forcing function $f(t)$. Explain that sentence.

Solution For a first order equation, an initial condition $y(0)$ is equivalent to adding $y(0)\delta(t)$ to the equation and starting that new equation at zero.

19 What is $\delta(t) * \delta(t)$? What is $\delta(t-1) * \delta(t-2)$? What is $\delta(t-1)$ *times* $\delta(t-2)$?

Solution $\delta(t) * \delta(t) = \delta(t)$

$\delta(t-1) * \delta(t-2) = \delta(t-3)$

$\delta(t-1)$ *times* $\delta(t-2)$ equals the zero function.

20 By Laplace transform, solve $y' = y$ with $y(0) = 1$ to find a very familiar $y(t)$.

Solution

$$\begin{aligned} y' &= y \qquad y(0) = 1 \\ sY(s) - 1 &= Y(s) \\ Y(s) &= \frac{1}{s - 1} \text{ gives } y(t) = \boldsymbol{e^t}. \end{aligned}$$

21 By Fourier transform as in (9), solve $-y'' + y =$ box function $b(x)$ on $0 \le x \le 1$.

Solution The Fourier transform of $-y'' + y = b(x)$ is

$$(k^2 + 1)\,\widehat{y}\,(k) = \widehat{b}(k) = \int_0^1 e^{-ikx}\,dx = \frac{1 - e^{-ik}}{ik}.$$

$$\widehat{y}(k) = \frac{1 - e^{-ik}}{(k^2 + 1)(ik)}$$

This transform must be inverted to find $\boldsymbol{y(x)}$. In reality I would solve separately on $x \le 0$ and $0 \le x \le 1$ and $x \ge 1$. Then matching at the breakpoints $x = 0$ and $x = 1$ determines the free constants in the separate solutions.

22 There is a big difference in the solutions to $y'' + By' + Cy = f(x)$, between the cases $B^2 < 4C$ and $B^2 > 4C$. Solve $y'' + y = \delta$ and $y'' - y = \delta$ with $y(\pm\infty) = 0$.

Solution (a) The delta function produces a unit jump in y' at $x = 0$:

$y'' + y = 0$ has $y = c_1 \cos x + c_2 \sin x$ for $x < 0$, $y = C_1 \sin x$ for $x > 0$.

The jump in y' gives $C_2 - c_2 = 1$. The condition on $y(\pm\,\infty)$ does not apply to this first equation.

$y'' - y = 0$ has $y = ce^x$ for $x < 0$ and $y = Ce^{-x}$ for $x > 0$; then $y(\pm\,\infty) = 0$.

Matching y at $x = 0$ gives $c = C$.

Jump in y' at $x = 0$ gives $-C - c = 1$ so $c = C = -\frac{1}{2}$

Solution $y(x) = -\frac{1}{2}e^x$ for $x \le 0$ and $y(x) = -\frac{1}{2}e^{-x}$ for $x \ge 0$

23 (*Review*) Why do the constant $f(t) = 1$ and the unit step $H(t)$ have the same Laplace transform $1/s$? Answer: Because the transform does not notice _____ .

Solution The Laplace Transform **does not notice any values of** $\boldsymbol{f(t)}$ **for** $\boldsymbol{t < 0}$.