DIFFERENTIAL EQUATIONS
AND
LINEAR ALGEBRA

MANUAL FOR INSTRUCTORS

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Problem Set 6.1, page 333

1 A has eigenvalues 1 and $\frac{1}{2}$, A^2 has eigenvalues 1 and $(\frac{1}{2})^2 = \frac{1}{4}$, A^∞ has eigenvalues 1 and 0 (notice $(\frac{1}{2})^\infty = 0$).

(a) Exchange the rows of A to get B:

$$B = \begin{bmatrix} .2 & .7 \\ .8 & .3 \end{bmatrix} \text{ has eigenvalues 1 and } -\tfrac{1}{2}.$$

B is still a Markov matrix, so $\lambda = 1$ is still an eigenvalue. The sum down the main diagonal (the "trace") is now .5 so the second eigenvalue must be $-.5$. Then trace $= .2 + .3 = 1 - .5$.

Zero eigenvalues remain zero after elimination because the matrix remains singular and its determinant remains zero.

2 A has $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $x_1 = (-2, 1)$ and $x_2 = (1, 1)$. The matrix $A + I$ has the same eigenvectors, with eigenvalues increased by 1 to **0** and **6**. That zero eigenvalue correctly indicates that $A + I$ is singular.

3 A has $\lambda_1 = 2$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1, 1)$ and $\boldsymbol{x}_2 = (2, -1)$. A^{-1} has the same eigenvectors, with eigenvalues $1/\lambda = \frac{1}{2}$ and -1.

4 A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace $= -1$ and determinant $= -6$) with $x_1 = (3, -2)$ and $x_2 = (1, 1)$. A^2 has the *same eigenvectors* as A, with eigenvalues $\lambda_1^2 = 9$ and $\lambda_2^2 = 4$.

5 A and B have eigenvalues 1 and 3. $A + B$ has $\lambda_1 = 3$, $\lambda_2 = 5$. Eigenvalues of $A + B$ *are not equal* to eigenvalues of A plus eigenvalues of B.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda = 2 \pm \sqrt{3}$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal*** (this is proved in section 6.6, Problems 18-19).

7 U is triangular so its eigenvalues are the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$. (This is because det $(U - \lambda I)$ will be just the product $(u_{11} - \lambda)(u_{22} - \lambda) \ldots (u_{nn} - \lambda)$ from the main diagonal.)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ with } \lambda = 2 \text{ and } 0 \qquad U = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } \lambda = 1 \text{ and } 0.$$

8 (a) Multiply $A\boldsymbol{x}$ to see $\lambda\boldsymbol{x}$ which reveals λ (b) Solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ to find $\boldsymbol{x}$.

9 (a) Multiply by A: $A(A\boldsymbol{x}) = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x}$ gives $A^2\boldsymbol{x} = \boldsymbol{\lambda}^2\boldsymbol{x}$ (b) Multiply by A^{-1}: $\boldsymbol{x} = A^{-1}A\boldsymbol{x} = A^{-1}\lambda\boldsymbol{x} = \lambda A^{-1}\boldsymbol{x}$ gives $A^{-1}\boldsymbol{x} = \frac{1}{\boldsymbol{\lambda}}\boldsymbol{x}$ (c) Add $I\boldsymbol{x} = \boldsymbol{x}$: $(A + I)\boldsymbol{x} = (\boldsymbol{\lambda} + \mathbf{1})\boldsymbol{x}$.

10 A has $\lambda_1 = 1$ and $\lambda_2 = .4$ with $\boldsymbol{x}_1 = (1, 2)$ and $\boldsymbol{x}_2 = (1, -1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (.4)^{100}$ which is near zero. So A^{100} is very near A^∞: same eigenvectors and close eigenvalues.

11 With $\lambda = 0, 1, 2$ the rank is **2**. The eigenvalues of B^2 are $0, 1, 4$. The eigenvalues of $(B^2 + I)^{-1}$ are $(0 + 1)^{-1} = 1, (1 + 1)^{-1} = \frac{1}{2}, (4 + 1)^{-1} = \frac{1}{5}$.

12 The projection matrix P has $\lambda = 1, 0, 1$ with eigenvectors $(1, 2, 0)$, $(2, -1, 0)$, $(0, 0, 1)$. Add the first and last vectors: $(1, 2, 1)$ also has $\lambda = 1$. *Note* $P^2 = P$ leads to $\lambda^2 = \lambda$ so $\lambda = 0$ or 1.

13 (a) $P\boldsymbol{u} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}) = \boldsymbol{u}$ so $\lambda = 1$ (b) $P\boldsymbol{v} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}) = \mathbf{0}$ (c) $\boldsymbol{x}_1 = (-1, 1, 0, 0)$, $\boldsymbol{x}_2 = (-3, 0, 1, 0)$, $\boldsymbol{x}_3 = (-5, 0, 0, 1)$ all have $P\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0}$.

14 Two eigenvectors of this rotation matrix are $\boldsymbol{x}_1 = (1, i)$ and $\boldsymbol{x}_2 = (1, -i)$ (more generally $c\boldsymbol{x}_1$, and $d\boldsymbol{x}_2$ with $cd \neq 0$).

15 These matrices all have $\lambda_1 = 0$ and $\lambda_2 = 0$ (which we can see from trace $= 0$ and determinant $= 0$):

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } A^2 = 0 \qquad A = \begin{bmatrix} a & -a \\ a & -a \end{bmatrix} \text{ has } A^2 = 0.$$

16 $\lambda = 0, 0, 6$ (*notice rank* 1 *and trace* 6) with $\boldsymbol{x}_1 = (0, -2, 1)$, $\boldsymbol{x}_2 = (1, -2, 0)$, $\boldsymbol{x}_3 = (1, 2, 1)$.

17 $\begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ so $\lambda_1 = 6$. Then $\lambda_2 = 1$ to make trace $= 5 + 2 = 6 + 1$.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

The other eigenvalue is $d - b$ to make trace $= a + d = (a + b) + (d - b)$.

18 These 3 matrices have $\lambda = 4$ and 5, trace 9, $\det 20$: $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -3 & 7 \end{bmatrix}$.

19 (a) $\boldsymbol{u}$ is a basis for the nullspace, $\boldsymbol{v}$ and $\boldsymbol{w}$ give a basis for the column space
(b) $\boldsymbol{x} = (0, \frac{1}{3}, \frac{1}{5})$ is a particular solution. Add any $c\boldsymbol{u}$ from the nullspace
(c) If $A\boldsymbol{x} = \boldsymbol{u}$ had a solution, $\boldsymbol{u}$ would be in the column space: wrong dimension 3.

20 (a) $A = \begin{bmatrix} 0 & -1 \\ \mathbf{-28} & \mathbf{11} \end{bmatrix}$ has trace 11 and determinant 28, so $\lambda = 4$ and 7.

(b) $A = \begin{bmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}$ has trace $\lambda_1 + \lambda_2$ and determinant $\lambda_1\lambda_2$ so its eigenvalues must be λ_1 and λ_2. This is a typical **companion matrix**.

21 $(A - \lambda I)$ has the same determinant as $(A - \lambda I)^{\mathrm{T}}$ because every square matrix has $\det M = \det M^{\mathrm{T}}$. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have *different eigenvectors*.

22 $\lambda = 1$ (for Markov), 0 (for singular), $-\frac{1}{2}$ (so sum of eigenvalues $=$ trace $= \frac{1}{2}$).

23 If you know n independent eigenvectors and their eigenvalues, you know the matrix A. In Section 6.2, the $\boldsymbol{x}$'s and $\boldsymbol{\lambda}$'s go into V and Λ, and the matrix must be $A = V\Lambda V^{-1}$. In this section, Problem 23 suggests that $A\boldsymbol{v} = B\boldsymbol{v}$ for every vector $\boldsymbol{v}$ (which proves $A = B$) because

$$\boldsymbol{v} = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n \qquad A\boldsymbol{v} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n = B\boldsymbol{v}.$$

24 The block matrix has $\lambda = 1, 2$ from B and $5, 7$ from D. All entries of C are multiplied by zeros in $\det(A - \lambda I)$, so C has no effect on the eigenvalues.

25 A has rank 1 with eigenvalues $0, 0, 0, 4$ (the 4 comes from the trace of A). C has rank 2 (ensuring two zero eigenvalues) and $(1, 1, 1, 1)$ is an eigenvector with $\lambda = 2$. With trace 4, the other eigenvalue is also $\lambda = 2$, and its eigenvector is $(1, -1, 1, -1)$.

26 B has $\lambda = -1, -1, -1, 3$ and C has $\lambda = 1, 1, 1, -3$. Both have $\det = -3$.

27 Triangular matrix: $\lambda(A) = 1, 4, 6$; $\lambda(B) = 2, \sqrt{3}, -\sqrt{3}$; Rank-1 matrix: $\lambda(C) = 0, 0, 6$.

28 $\det \begin{bmatrix} 0-\lambda & 1 & 0 \\ 0 & 0-\lambda & 1 \\ 1 & 0 & 0-\lambda \end{bmatrix} = -\lambda^3 + 1 = 0$ for $\lambda = 1, e^{2\pi i/3}, e^{-2\pi i/3}$.

Those complex eigenvalues λ_2, λ_3 are $\cos 120° \pm i \sin 120° = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

The trace of P is $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

$\det \begin{bmatrix} 0-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 0-\lambda \end{bmatrix} = -\lambda^3 + \lambda^2 + \lambda - 1 = 0$ for $\lambda = 1, 1, -1$. The trace is $1 + 1 - 1 = \mathbf{1}$. Three eigenvectors are $(1, 1, 1)$ and $(1, 0, 1)$ and $(1, 0, -1)$. Since P is symmetric we could have chosen orthogonal eigenvectors—change the first to $(0, 1, 0)$.

29 Set $\lambda = 0$ in $\det(A - \lambda I) = (\lambda_1 - \lambda) \ldots (\lambda_n - \lambda)$ to find $\det A = (\lambda_1)(\lambda_2) \cdots (\lambda_n)$.

30 $\lambda_1 = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc})$ and $\lambda_2 = \frac{1}{2}(a + d - \sqrt{})$ add to $a + d$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = (\lambda - 3)(\lambda - 4) = \lambda^2 - 7\lambda + 12$.

Problem Set 6.2, page 345

Questions 1–7 are about the eigenvalue and eigenvector matrices Λ and V.

1 (a) Factor these two matrices into $A = V\Lambda V^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

(b) If $A = V\Lambda V^{-1}$ then $A^3 = (V)(\Lambda^3)(V^{-1})$ and $A^{-1} = (V)(\Lambda^{-1})(V^{-1})$.

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

2 If A has $\lambda_1 = 2$ with eigenvector $\boldsymbol{x}_1 = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$ and $\lambda_2 = 5$ with $\boldsymbol{x}_2 = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$, use $V\Lambda V^{-1}$ to find A. No other matrix has the same λ's and $\boldsymbol{x}$'s.

Put the eigenvectors in V and eigenvalues in Λ. $A = V\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

3 Suppose $A = V\Lambda V^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (V)(\Lambda + 2I)(V)^{-1}$.

If $A = V\Lambda V^{-1}$ then the eigenvalue matrix for $A + 2I$ is $\Lambda + 2I$ and the eigenvector matrix is still V. $\quad V(\Lambda + 2I)V^{-1} = V\Lambda V^{-1} + V(2I)V^{-1} = A + 2I$.

4 True or false: If the columns of V (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) V is invertible (d) V is diagonalizable.

(a) False: don't know λ's (b) True (c) True (d) False: need eigenvectors of V

5 If the eigenvectors of A are the columns of I, then A is a _____ matrix. If the eigenvector matrix V is triangular, then V^{-1} is triangular. Prove that A is also triangular.

With $V = I, A = V\Lambda V^{-1} = \Lambda$ is a diagonal matrix. If V is triangular, then V^{-1} is triangular, so $V\Lambda V^{-1}$ is also triangular.

6 Describe all matrices V that diagonalize this matrix A (find all eigenvectors):

$$A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}.$$

Then describe all matrices that diagonalize A^{-1}.

The columns of V are nonzero multiples of $(2,1)$ and $(0,1)$: in either order. The same matrices V will diagonize A^{-1}.

7 Write down the most general matrix that has eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$A = V\Lambda V^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /2 = \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix} /2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ for any a and b.

Questions 8–10 are about Fibonacci and Gibonacci numbers.

8 Diagonalize the Fibonacci matrix by completing V^{-1}:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}.$$

Do the multiplication $V\Lambda^k V^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to find its second component. This is the kth Fibonacci number $F_k = \left(\lambda_1^k - \lambda_2^k\right) / \left(\lambda_1 - \lambda_2\right)$.

$A = V\Lambda V^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$. $V\Lambda^k V^{-1} = \dfrac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \textit{2nd component} \text{ is } F_k \\ (\lambda_1^k - \lambda_2^k)/(\lambda_1 - \lambda_2) \end{bmatrix}$.

9 Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2} G_{k+1} + \tfrac{1}{2} G_k \\ G_{k+1} &= G_{k+1} \end{aligned} \qquad \text{is} \qquad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find A and its eigenvalues and eigenvectors.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = V\Lambda^n V^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

(a) $A = \begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$ has $\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2}$ with $\boldsymbol{x}_1 = (1,1),\ \boldsymbol{x}_2 = (1,-2)$

(b) $A^n = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \to A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

10 Prove that every third Fibonacci number in $0, 1, 1, 2, 3, \ldots$ is even.

The rule $F_{k+2} = F_{k+1} + F_k$ produces the pattern: even, odd, odd, even, odd, odd, . . .

Questions 11–14 are about diagonalizability.

11 True or false : If the eigenvalues of A are $2, 2, 5$ then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

(a) *True* (no zero eigenvalues) (b) *False* (repeated $\lambda = 2$ may have only one line of eigenvectors) (c) *False* (repeated λ may have a full set of eigenvectors)

12 True or false : If the only eigenvectors of A are multiples of $(1, 4)$ then A has

(a) no inverse (b) a repeated eigenvalue (c) no diagonalization $V \Lambda V^{-1}$.

(a) False: don't know λ (b) True: an eigenvector is missing (c) True.

13 Complete these matrices so that $\det A = 25$. Then check that $\lambda = 5$ is repeated—the trace is 10 so the determinant of $A - \lambda I$ is $(\lambda - 5)^2$. Find an eigenvector with $A\boldsymbol{x} = 5\boldsymbol{x}$. These matrices will not be diagonalizable because there is no second line of eigenvectors.

$$A = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 9 & 4 \\ & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 10 & 5 \\ -5 & \end{bmatrix}$$

$A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ (or other), $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$; only eigenvectors are $\boldsymbol{x} = (c, -c)$.

14 The matrix $A = \left[\begin{smallmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} \end{smallmatrix}\right]$ is not diagonalizable because the rank of $A - 3I$ is _____. Change one entry to make A diagonalizable. Which entries could you change ?

The rank of $A - 3I$ is $r = 1$. Changing any entry except $a_{12} = 1$ makes A diagonalizable (A will have unequal eigenvalues, so eigenvectors are independent.)

Questions 15–19 are about powers of matrices.

15 $A^k = V \Lambda^k V^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than _____. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

$A^k = V \Lambda^k V^{-1}$ approaches zero **if and only if every** $|\boldsymbol{\lambda}| < \mathbf{1}$; $A_1^k \to A_1^\infty$, $A_2^k \to 0$.

16 (Recommended) Find Λ and V to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $V\Lambda^k V^{-1}$? In the columns of this limiting matrix you see the _____.

$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $V\Lambda^k V^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: *steady state*.

17 Find Λ and V to diagonalize A_2 in Problem 15. What is $(A_2)^{10}\boldsymbol{u}_0$ for these $\boldsymbol{u}_0$?

$$\boldsymbol{u}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}_0 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

$\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$, $S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $A_2^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A_2^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $A_2^{10} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ because $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ is the sum of $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

18 Diagonalize A and compute $V\Lambda^k V^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \text{has} \qquad A^k = \frac{1}{2} \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $A^k = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Multiply those last three matrices to get $A^k = \frac{1}{2} \begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$.

19 Diagonalize B and compute $V\Lambda^k V^{-1}$ to prove this formula for B^k:

$$B = \begin{bmatrix} 5 & 1 \\ 0 & 4 \end{bmatrix} \qquad \text{has} \qquad B^k = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}.$$

$B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5^k & 5^k - 4^k \\ 0 & 4^k \end{bmatrix}$.

20 Suppose $A = V\Lambda V^{-1}$. Take determinants to prove $\det A = \det \Lambda = \lambda_1 \lambda_2 \cdots \lambda_n$. This quick proof only works when A can be _____.

$\det A = (\det V)(\det \Lambda)(\det V^{-1}) = \det \Lambda = \lambda_1 \cdots \lambda_n$. This proof works when A is *diagonalizable*.

21 Show that trace VT = trace TV, by adding the diagonal entries of VT and TV:

$$V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad T = \begin{bmatrix} q & r \\ s & t \end{bmatrix}.$$

Choose T as ΛV^{-1}. Then $V\Lambda V^{-1}$ has the same trace as $\Lambda V^{-1} V = \Lambda$. The trace of A equals the trace of Λ, which is certainly the sum of the eigenvalues.

trace $VT = (aq + bs) + (cr + dt)$ is equal to $(qa + rc) + (sb + td) =$ trace TV. Diagonalizable trace of $V\Lambda V^{-1}$ = trace of $(\Lambda V^{-1})V$ = trace of Λ: *sum of the* λ's.

22 $AB - BA = I$ is impossible since the left side has trace = _____. But find an elimination matrix so that $A = E$ and $B = E^{\mathrm{T}}$ give

$$AB - BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{which has trace zero.}$$

$AB - BA = I$ is impossible since trace AB − trace $BA = zero \neq$ trace I. $AB - BA = C$ is possible when trace $(C) = 0$.
$E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ has $EE^{\mathrm{T}} - E^{\mathrm{T}}E = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

23 If $A = V\Lambda V^{-1}$, diagonalize the block matrix $B = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{2A} \end{bmatrix}$. Find its eigenvalue and eigenvector (block) matrices.

If $A = V\Lambda V^{-1}$ then $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix} \begin{bmatrix} V^{-1} & 0 \\ 0 & V^{-1} \end{bmatrix}$. So B has the additional eigenvalues $2\lambda_1, \ldots, 2\lambda_n$.

24 Consider all 4 by 4 matrices A that are diagonalized by the same fixed eigenvector matrix V. Show that the A's form a subspace (cA and $A_1 + A_2$ have this same V). What is this subspace when $V = I$? What is its dimension ?

The A's form a subspace since cA and $A_1 + A_2$ all have the same V. When $V = I$ the A's with those eigenvectors give the subspace of diagonal matrices. Dimension 4.

25 Suppose $A^2 = A$. On the left side A multiplies each column of A. Which of our four subspaces contains eigenvectors with $\lambda = 1$? Which subspace contains eigenvectors with $\lambda = 0$? From the dimensions of those subspaces, A has a full set of independent eigenvectors. So every matrix with $A^2 = A$ can be diagonalized.

If A has columns $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ then column by column, $A^2 = A$ means every $A\boldsymbol{x}_i = \boldsymbol{x}_i$. All vectors in the column space (combinations of those columns $\boldsymbol{x}_i$) are eigenvectors with $\lambda = 1$. Always the nullspace has $\lambda = 0$ (A might have dependent columns, so there could be less than n eigenvectors with $\lambda = 1$). Dimensions of those spaces add to n by the Fundamental Theorem, so A *is diagonalizable* (n independent eigenvectors altogether).

26 (Recommended) Suppose $A\boldsymbol{x} = \lambda\boldsymbol{x}$. If $\lambda = 0$ then $\boldsymbol{x}$ is in the nullspace. If $\lambda \neq 0$ then $\boldsymbol{x}$ is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors ?

Two problems: The nullspace and column space can overlap, so $\boldsymbol{x}$ could be in both. There may not be r independent eigenvectors in the column space.

27 The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9 :

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = V\sqrt{\Lambda}\,V^{-1}$. Why is there no real matrix square root of B ?

$R = V\sqrt{\Lambda}V^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$. $\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, trace is not real.
Note that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ can have $\sqrt{-1} = i$ and $-i$, trace 0, real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

28 The powers A^k approach zero if all $|\lambda_i| < 1$ and they blow up if any $|\lambda_i| > 1$. Peter Lax gives these striking examples in his book *Linear Algebra*:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 6.9 \\ -3 & -4 \end{bmatrix}$$

$$\|\boldsymbol{A^{1024}}\| > \boldsymbol{10^{700}} \quad \boldsymbol{B^{1024} = I} \quad \boldsymbol{C^{1024} = -C} \quad \|\boldsymbol{D^{1024}}\| < \boldsymbol{10^{-78}}$$

Find the eigenvalues $\lambda = e^{i\theta}$ of B and C to show $B^4 = I$ and $C^3 = -I$.

B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1 and $B^4 = I$. C has $\lambda = (1 \pm \sqrt{3}i)/2$. This is $\exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ and $C^{1024} = -C$.

29 If A and B have the same λ's with the same full set of independent eigenvectors, their factorizations into _____ are the same. So $A = B$.

The factorizations of A and B into $V\Lambda V^{-1}$ are the same. So $A = B$. (This is the same as Problem 6.1.25, expressed in matrix form.)

30 Suppose the same V diagonalizes both A and B. They have the same eigenvectors in $A = V\Lambda_1 V^{-1}$ and $B = V\Lambda_2 V^{-1}$. Prove that $AB = BA$.

$A = V\Lambda_1 V^{-1}$ and $B = V\Lambda_2 V^{-1}$. Diagonal matrices always give $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$. Then $AB = BA$ from $V\Lambda_1 V^{-1}V\Lambda_2 V^{-1} = V\boldsymbol{\Lambda_1\Lambda_2}V^{-1} = V\boldsymbol{\Lambda_2\Lambda_1}V^{-1}$. This is $V\Lambda_2 V^{-1}V\Lambda_1 V^{-1} = BA$.

31 (a) If $A = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{0} & \mathbf{d} \end{bmatrix}$ then the determinant of $A - \lambda I$ is $(\lambda - a)(\lambda - d)$. Check the "Cayley-Hamilton Theorem" that $(A - aI)(A - dI) =$ *zero matrix*.

(b) Test the Cayley-Hamilton Theorem on Fibonacci's $A = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$. The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.

(a) $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has $\lambda = a$ and $\lambda = d$: $(A-aI)(A-dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix} \begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 - A - I = 0$ is true, matching $\lambda^2 - \lambda - 1 = 0$ as the Cayley-Hamilton Theorem predicts.

32 Substitute $A = V\Lambda V^{-1}$ into the product $(A-\lambda_1 I)(A-\lambda_2 I)\cdots(A-\lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \det(A - \lambda I)$. The ***Cayley-Hamilton Theorem*** says that this product is always $p(A) =$ *zero matrix*, even if A is not diagonalizable.

When $A = V\Lambda V^{-1}$ is diagonalizable, the matrix $A-\lambda_j I = V(\Lambda-\lambda_j I)V^{-1}$ will have 0 in the j, j diagonal entry of $\Lambda - \lambda_j I$. In the product $p(A) = (A-\lambda_1 I)\cdots(A-\lambda_n I)$, each inside V^{-1} cancels V. This leaves V times (*product of diagonal matrices* $\Lambda-\lambda_j I$) times V^{-1}. That product is the zero matrix because the factors produce a zero in each diagonal position. Then $p(A) =$ zero matrix, which is the Cayley-Hamilton Theorem. (If A is not diagonalizable, one proof is to take a sequence of diagonalizable matrices approaching A.)

Comment I have also seen the following reasoning but I am not convinced:

Apply the formula $AC^{\mathrm{T}} = (\det A)I$ from Section 5.3 to $A - \lambda I$ with variable λ. Its cofactor matrix C will be a polynomial in λ, since cofactors are determinants:

$$(A - \lambda I)\ \text{cof}\ (A - \lambda I)^{\mathrm{T}} = \det(A - \lambda I)I = p(\lambda)I.$$

"For fixed A, this is an identity between two matrix polynomials." Set $\lambda = A$ to find the zero matrix on the left, so $p(A) =$ zero matrix on the right—which is the Cayley-Hamilton Theorem.

I am not certain about the key step of substituting a matrix for λ. If other matrices B are substituted, does the identity remain true? If $AB \neq BA$, even the order of multiplication seems unclear . . .

Challenge Problems

33 The nth power of rotation through θ is rotation through $n\theta$:

$$A^n = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Prove that neat formula by diagonalizing $A = V\Lambda V^{-1}$. The eigenvectors (columns of V) are $(1, i)$ and $(i, 1)$. You need to know Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

The eigenvalues of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ are $\lambda = e^{i\theta}$ and $e^{-i\theta}$ (trace $2\cos\theta$ and $\det = 1$). Their eigenvectors are $(1, -i)$ and $(1, i)$:

$$A^n = V\Lambda^n V^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{in\theta} & \\ & e^{-in\theta} \end{bmatrix} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} /2i$$

$$= \begin{bmatrix} (e^{in\theta} + e^{-in\theta})/2 & \cdots \\ (e^{in\theta} - e^{-in\theta})/2i & \cdots \end{bmatrix} = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}.$$

Geometrically, n rotations by θ give one rotation by $n\theta$.

34 The transpose of $A = V\Lambda V^{-1}$ is $A^{\mathrm{T}} = (V^{-1})^{\mathrm{T}}\Lambda V^{\mathrm{T}}$. The eigenvectors in $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{y}$ are the columns of that matrix $(V^{-1})^{\mathrm{T}}$. They are often called ***left eigenvectors***.

How do you multiply three matrices $V\Lambda V^{-1}$ to find this formula for A?

$$\textbf{Sum of rank-1 matrices} \quad A = V\Lambda V^{-1} = \lambda_1 \boldsymbol{x}_1 \boldsymbol{y}_1^{\mathrm{T}} + \cdots + \lambda_n \boldsymbol{x}_n \boldsymbol{y}_n^{\mathrm{T}}.$$

Columns of V times rows of ΛV^{-1} will give r rank-1 matrices ($r =$ rank of A).

35 The inverse of $A = \textbf{eye}(n) + \textbf{ones}(n)$ is $A^{-1} = \textbf{eye}(n) + C * \textbf{ones}(n)$. Multiply AA^{-1} to find that number C (depending on n).

Note that $\mathsf{ones}(n) * \mathsf{ones}(n) = n * \mathsf{ones}(n)$. This leads to $C = 1/(n+1)$.

$$\begin{aligned} AA^{-1} &= (\mathsf{eye}(n) + \mathsf{ones}(n)) * (\mathsf{eye}(n) + C * \mathsf{ones}(n)) \\ &= \mathsf{eye}(n) + (1 + C + Cn) * \mathsf{ones}(n) = \mathsf{eye}(n). \end{aligned}$$

Problem Set 6.3, page 357

1 Find all solutions $\boldsymbol{y} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ to $\boldsymbol{y'} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}\boldsymbol{y}$. Which solution starts from $\boldsymbol{y}(0) = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = (2,2)$?

The eigenvalues come from $\det(A - \lambda I) = 0$. This is

$$\lambda^2 - 8\lambda + 12 = (\lambda - 2)(\lambda - 6) = 0 \text{ so } \lambda = \mathbf{2, 6}$$

Eigenvectors: $(A - 2I)\boldsymbol{x}_1 = \mathbf{0}$ and $(A - 6I)\boldsymbol{x}_2 = 0$ give $\boldsymbol{x}_1 = (1, -1)$ and $\boldsymbol{x}_2 = (1, 3)$

Solutions are $\boldsymbol{y}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Constants c_1, c_2 come from $\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \boldsymbol{y}(\mathbf{0}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Then $\boldsymbol{c_1} = \boldsymbol{c_2} = \mathbf{1}$.

2 Find two solutions of the form $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ to $\boldsymbol{y'} = \begin{bmatrix} 3 & 10 \\ 2 & 4 \end{bmatrix}\boldsymbol{y}$.

The eigenvalues come from $\lambda^2 - 7\lambda - 8 = 0$. Factor into $(\lambda - 8)(\lambda + 1)$ to see $\lambda = \mathbf{8}$, and $\mathbf{-1}$.

$$(A - 8I)\boldsymbol{x}_1 = \begin{bmatrix} -5 & 10 \\ 2 & -1 \end{bmatrix}\boldsymbol{x}_1 = \mathbf{0} \text{ gives } \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A + I)\boldsymbol{x}_2 = \begin{bmatrix} 4 & 10 \\ 2 & 5 \end{bmatrix}\boldsymbol{x}_2 = \mathbf{0} \text{ gives } \boldsymbol{x}_2 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

The two solutions are $y(t) = e^{8t}\boldsymbol{x}_1$ and $e^{-t}\boldsymbol{x}_2$

3 If $a \neq d$, find the eigenvalues and eigenvectors and the complete solution to $\boldsymbol{y}' = A\boldsymbol{y}$. This equation is stable when a and d are _____.

$$\boldsymbol{y'} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\boldsymbol{y}.$$

The eigenvalues are $\lambda = a$ and $\lambda = d$. The eigenvectors come from

$$(A - aI)\,\boldsymbol{x}_1 = \begin{bmatrix} 0 & b \\ 0 & d - a \end{bmatrix}\boldsymbol{x}_1 = \mathbf{0}. \quad \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - dI)\,\boldsymbol{x}_2 = \begin{bmatrix} a - d & b \\ 0 & 0 \end{bmatrix}\boldsymbol{x}_2 = \mathbf{0}. \quad \boldsymbol{x}_2 = \begin{bmatrix} b \\ d - a \end{bmatrix}$$

Two solutions are $y = e^{at}\boldsymbol{x}_1$ and $y = e^{dt}\boldsymbol{x}_2$. Stability for **negative** a and d.

4 If $a \neq -b$, find the solutions $e^{\lambda_1 t}\boldsymbol{x}_1$ and $e^{\lambda_2 t}\boldsymbol{x}_2$ to $y' = A\boldsymbol{y}$:

$$A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}. \quad \text{Why is } \boldsymbol{y'} = A\boldsymbol{y} \text{ not stable?}$$

A is singular so $\lambda_1 = 0$. Trace is $a + b$ so $\lambda_2 = a + b$. $(A - 0I)\,\boldsymbol{x}_1 = \mathbf{0}$ gives

$\boldsymbol{x}_1 = \begin{bmatrix} b \\ -a \end{bmatrix}$ $\quad (A - (a + b)I)\,\boldsymbol{x}_2 = \begin{bmatrix} -b & b \\ a & -a \end{bmatrix}\boldsymbol{x}_2 = 0$ gives $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The system is not stable because $\lambda = 0$ is an eigenvalue. If $\lambda_2 = a + b$ is negative, the system is "neutral" and the solution approaches a steady state (a multiple of $\boldsymbol{x}_1$).

5 Find the eigenvalues λ_1, λ_2, λ_3 and the eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ of A. Write $\boldsymbol{y}(0) = (0, 1, 0)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3 = V\boldsymbol{c}$ and solve $\boldsymbol{y}' = A\boldsymbol{y}$. What is the limit of $\boldsymbol{y}(t)$ as $t \to \infty$ (the steady state)? *Steady states come from* $\lambda = 0$.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Calculation gives $\det(A - \lambda I) = -(\lambda + 1)\lambda(\lambda + 3)$ and eigenvalues $\lambda = 0, -1, -3$.

$$\lambda=0 \text{ has eigenvector } \boldsymbol{x}_1=\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \lambda=-1 \text{ has } \boldsymbol{x}_2=\begin{bmatrix}1\\0\\-1\end{bmatrix} \quad \lambda=-3 \text{ has } \boldsymbol{x}_3=\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

Notice: Those eigenvectors are orthogonal (because A is symmetric). Then $\boldsymbol{y}(0)$ is

$$(0,1,0) = \frac{1}{3}(\boldsymbol{x}_1 - \boldsymbol{x}_3) \text{ so } \boldsymbol{y}(t) = \frac{1}{3}e^{0t}\boldsymbol{x}_1 - \frac{1}{3}e^{-3t}\boldsymbol{x}_2 \text{ approaches } y(\infty) = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

6 The simplest 2 by 2 matrix without two independent eigenvectors has $\lambda = 0, 0$:

$$\begin{bmatrix}y_1\\y_2\end{bmatrix}' = A\boldsymbol{y} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} \text{ has a first solution } \begin{bmatrix}y_1\\y_2\end{bmatrix} = e^{0t}\begin{bmatrix}1\\0\end{bmatrix}.$$

Find a second solution to these equations ${y_1}' = y_2$ and ${y_2}' = 0$. That second solution starts with t times the first solution to give $y_1 = t$. What is y_2?

Note A complete discussion of $\boldsymbol{y'} = A\boldsymbol{y}$ for all cases of repeated λ's would involve the *Jordan form* of A: too technical. Section 6.4 shows that a triangular form is sufficient, as Problems 6 and 8 confirm. We can solve for y_2 and then y_1.

The first solution to $y_1' = y_2$ and $y_2' = 0$ is $(y_1(t), y_2(t)) = (1, 0) =$ eigenvector.
A second solution has $(y_1, y_2) = (t, 1)$. The factor t appears when there is no $\boldsymbol{x}_2$.

7 Find two λ's and $\boldsymbol{x}$'s so that $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ solves

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix}4 & 3\\0 & 1\end{bmatrix}\boldsymbol{y}.$$

What combination $\boldsymbol{y} = c_1e^{\lambda_1 t}\boldsymbol{x}_1 + c_2e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{y}(0) = (5, -2)$?

$\boldsymbol{y}_1 = e^{4t}\begin{bmatrix}1\\0\end{bmatrix}$, $\boldsymbol{y}_2 = e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$. If $\boldsymbol{y}(0) = \begin{bmatrix}5\\-2\end{bmatrix}$, then $\boldsymbol{y}(t) = 3e^{4t}\begin{bmatrix}1\\0\end{bmatrix} + 2e^{t}\begin{bmatrix}1\\-1\end{bmatrix}$.

8 Solve Problem 7 for $\boldsymbol{y} = (y, z)$ by back substitution, z before y:

$$\text{Solve } \frac{dz}{dt} = z \text{ from } z(0) = -2. \text{ Then solve } \frac{dy}{dt} = 4y + 3z \text{ from } y(0) = 5.$$

The solution for y will be a combination of e^{4t} and e^t. $\lambda = 4$ and 1. $z(t) = -2e^t$.

Then $dy/dt = 4y - 6e^t$ with $y(0) = 5$ gives $y(t) = 3e^{4t} + 2e^t$ as in Problem 7.

9 (a) If every column of A adds to zero, why is $\lambda = 0$ an eigenvalue?

(b) With negative diagonal and positive off-diagonal adding to zero, $\boldsymbol{y}' = A\boldsymbol{y}$ will be a "continuous" Markov equation. Find the eigenvalues and eigenvectors, and the *steady state* as $t \to \infty$:

$$\text{Solve} \quad \frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix} \boldsymbol{y} \quad \text{with} \quad \boldsymbol{y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}. \quad \text{What is } \boldsymbol{y}(\infty)\,?$$

(a) If every column of A adds to zero, this means that the rows add to the zero row. So the rows are dependent, and A is singular, and $\lambda = 0$ is an eigenvalue.

(b) The eigenvalues of $A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$ are $\lambda_1 = 0$ with eigenvector $\boldsymbol{x}_1 = (3, 2)$ and $\lambda_2 = -5$ (to give trace $= -5$) with $\boldsymbol{x}_2 = (1, -1)$. Then the usual 3 steps:

1. Write $\boldsymbol{y}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ as $\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \boldsymbol{x}_1 + \boldsymbol{x}_2$
2. Follow those eigenvectors by $e^{0t}\boldsymbol{x}_1$ and $e^{-5t}\boldsymbol{x}_2$
3. The solution $\boldsymbol{y}(t) = \boldsymbol{x}_1 + e^{-5t}\boldsymbol{x}_2$ has steady state $\boldsymbol{x}_1 = (3, 2)$.

10 A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $d\boldsymbol{y}/dt = A\boldsymbol{y}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

$d(v + w)/dt = (w - v) + (v - w) = 0$, so the total $v + w$ is constant. $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

has $\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = -2 \end{matrix}$ with $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\begin{matrix} v(1) = 20 + 10e^{-2} & v(\infty) = 20 \\ w(1) = 20 - 10e^{-2} & w(\infty) = 20 \end{matrix}$

11 Reverse the diffusion of people in Problem 10 to $d\boldsymbol{z}/dt = -A\boldsymbol{z}$:

$$\frac{dv}{dt} = v - w \quad \text{and} \quad \frac{dw}{dt} = w - v.$$

The total $v + w$ still remains constant. How are the λ's changed now that A is changed to $-A$? But show that $v(t)$ grows to infinity from $v(0) = 30$.

$\dfrac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has $\lambda = 0$ and $+2$: $v(t) = 20 + 10e^{2t} \to \infty$ as $t \to \infty$.

12 A has real eigenvalues but B has complex eigenvalues:

$$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix} \quad B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix} \quad (a \text{ and } b \text{ are real})$$

Find the stability conditions on a and b so that all solutions of $d\boldsymbol{y}/dt = A\boldsymbol{y}$ and $d\boldsymbol{z}/dt = B\boldsymbol{z}$ approach zero as $t \to \infty$.

$A = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$ has real eigenvalues $a+1$ and $a-1$. These are both negative if $\boldsymbol{a < -1}$, and the solutions of $\boldsymbol{y}' = A\boldsymbol{y}$ approach zero. $B = \begin{bmatrix} b & -1 \\ 1 & b \end{bmatrix}$ has complex eigenvalues $b+i$ and $b-i$. These have negative real parts if $\boldsymbol{b < 0}$, and all solutions of $\boldsymbol{z}' = B\boldsymbol{z}$ approach zero.

13 Suppose P is the projection matrix onto the $45°$ line $y = x$ in $\mathbf{R}^2$. Its eigenvalues are 1 and 0 with eigenvectors $(1,1)$ and $(1,-1)$. If $d\boldsymbol{y}/dt = -P\boldsymbol{y}$ (notice minus sign) can you find the limit of $\boldsymbol{y}(t)$ at $t = \infty$ starting from $\boldsymbol{y}(0) = (3,1)$?

A projection matrix has eigenvalues $\lambda = 1$ and $\lambda = 0$. Eigenvectors $P\boldsymbol{x} = \boldsymbol{x}$ fill the subspace that P projects onto: here $\boldsymbol{x} = (1,1)$. Eigenvectors $P\boldsymbol{x} = \mathbf{0}$ fill the perpendicular subspace: here $\boldsymbol{x} = (1,-1)$. For the solution to $\boldsymbol{y}' = -P\boldsymbol{y}$,

$$\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \boldsymbol{y}(t) = e^{-t}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + e^{0t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ approaches } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

14 The rabbit population shows fast growth (from $6r$) but loss to wolves (from $-2w$). The wolf population always grows in this model ($-w^2$ would control wolves):

$$\frac{dr}{dt} = 6r - 2w \qquad \text{and} \qquad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectors. If $r(0) = w(0) = 30$ what are the populations at time t? After a long time, what is the ratio of rabbits to wolves?

$\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ has $\lambda_1 = 5,\ \boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \lambda_2 = 2,\ \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches $20/10$; e^{5t} dominates.

15 (a) Write $(4,0)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$ of these two eigenvectors of A:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = i\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = -i\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

(b) The solution to $d\boldsymbol{y}/dt = A\boldsymbol{y}$ starting from $(4,0)$ is $c_1e^{it}\boldsymbol{x}_1 + c_2e^{-it}\boldsymbol{x}_2$. Substitute $e^{it} = \cos t + i\sin t$ and $e^{-it} = \cos t - i\sin t$ to find $\boldsymbol{y}(t)$.

(a) $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ i \end{bmatrix} + 2\begin{bmatrix} 1 \\ -i \end{bmatrix}$. (b) Then $\boldsymbol{y}(t) = 2e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + 2e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 4\cos t \\ 4\sin t \end{bmatrix}$.

Questions 16–19 reduce second-order equations to first-order systems for (y, y').

16 Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $\boldsymbol{y} = (y, y')$:

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{y}.$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$. $A = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$ has $\det(A - \lambda I) = \lambda^2 - 5\lambda - 4 = 0$. Directly substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$ also gives $\lambda^2 = 5\lambda + 4$ and the same two values of λ. Those values are $\lambda = \frac{1}{2}(5 \pm \sqrt{41})$ by the quadratic formula.

17 Substitute $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ to show that $\lambda = 3$ is a repeated root. This is trouble; we need a second solution after e^{3t}. The matrix equation is

$$\frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}.$$

Show that this matrix has $\lambda = 3, 3$ and only one line of eigenvectors. *Trouble here too*. Show that the second solution to $y'' = 6y' - 9y$ is $y = te^{3t}$.

$A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with *one* independent eigenvector $(1, 3)$.

18 (a) Write down two familiar functions that solve the equation $d^2y/dt^2 = -9y$. Which one starts with $y(0) = 3$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -9y$ produces a vector equation $\boldsymbol{y}' = A\boldsymbol{y}$:

$$\boldsymbol{y} = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{y}.$$

Find $\boldsymbol{y}(t)$ by using the eigenvalues and eigenvectors of A: $\boldsymbol{y}(0) = (3, 0)$.

(a) $y(t) = \cos 3t$ and $\sin 3t$ solve $y'' = -9y$. It is $\mathbf{3}\cos 3t$ that starts with $y(0) = 3$ and $y'(0) = 0$. (b) $A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$ has det $= 9$: $\lambda = 3i$ and $-3i$ with $\boldsymbol{x} = (1, 3i)$ and $(1, -3i)$. Then $\boldsymbol{y}(t) = \frac{3}{2}e^{3it}\begin{bmatrix} 1 \\ 3i \end{bmatrix} + \frac{3}{2}e^{-3it}\begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} 3\cos 3t \\ -9\sin 3t \end{bmatrix}$.

19 If c is not an eigenvalue of A, substitute $\boldsymbol{y} = e^{ct}\boldsymbol{v}$ and find a particular solution to $d\boldsymbol{y}/dt = A\boldsymbol{y} - e^{ct}\boldsymbol{b}$. How does it break down when c is an eigenvalue of A?

Substituting $\boldsymbol{y} = e^{ct}\boldsymbol{v}$ gives $ce^{ct}\boldsymbol{v} = Ae^{ct}\boldsymbol{v} - e^{ct}\boldsymbol{b}$ or $(A - cI)\boldsymbol{v} = \boldsymbol{b}$ or $\boldsymbol{v} = (A - cI)^{-1}\boldsymbol{b}$ = particular solution. If c is an eigenvalue then $A - cI$ is not invertible.

20 A particular solution to $d\boldsymbol{y}/dt = A\boldsymbol{y} - \boldsymbol{b}$ is $\boldsymbol{y}_p = A^{-1}\boldsymbol{b}$, if A is invertible. The usual solutions to $d\boldsymbol{y}/dt = A\boldsymbol{y}$ give $\boldsymbol{y}_n$. Find the complete solution $\boldsymbol{y} = \boldsymbol{y}_p + \boldsymbol{y}_n$:

(a) $\frac{dy}{dt} = y - 4$ (b) $\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\boldsymbol{y} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

$\boldsymbol{y}_p = 4$ and $\boldsymbol{y}(t) = ce^t + 4$; $\boldsymbol{y}_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{y}(t) = c_1e^t\begin{bmatrix} 1 \\ t \end{bmatrix} + c_2e^t\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

21 Find a matrix A to illustrate each of the unstable regions in the stability picture:

(a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. These show the unstable cases

(a) $\lambda_1 < 0$ and $\lambda_2 > 0$ (b) $\lambda_1 > 0$ and $\lambda_2 > 0$ (c) $\lambda = a \pm ib$ with $a > 0$

22 Which of these matrices are stable ? Then Re $\lambda < 0$, trace < 0, and det > 0.

$$A_1 = \begin{bmatrix} -2 & -3 \\ -4 & -5 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -3 & -6 \end{bmatrix} \quad A_3 = \begin{bmatrix} -1 & 2 \\ -3 & -6 \end{bmatrix}.$$

A_1 is unstable (trace $= -7$ but determinant $= -2$; $\lambda_1 < 0$ but $\lambda_2 > 0$).

A_2 is unstable (singular so $\lambda_1 = 0$).

A_3 is stable (trace $= -7$ and determinant 12; $\lambda_1 < 0$ and $\lambda_2 < 0$).

23 For an n by n matrix with trace $(A) = T$ and det $(A) = D$, find the trace and determinant of $-A$. Why is $\boldsymbol{z}' = -A\boldsymbol{z}$ unstable whenever $\boldsymbol{y}' = A\boldsymbol{y}$ is stable ?

If trace $(A) = T$ then trace $(-A) = -T$

If determinant $(A) = D$ then determinant $(-A) = (-1)^n D$

The eigenvalues of $-A$ are $-$(eigenvalues of A).

24 (a) For a real 3 by 3 matrix with stable eigenvalues (Re $\lambda < 0$), show that trace < 0 and det < 0. Either three real negative λ or else $\lambda_2 = \overline{\lambda_1}$ and λ_3 is real.

(b) The trace and determinant of a 3 by 3 matrix do not determine all three eigenvalues ! Show that A is unstable even with trace < 0 and determinant < 0 :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -5 \end{bmatrix}.$$

(a) If all three real parts are negative (stability), trace $=$ sum of real parts < 0.

Also $\det = \lambda_1\lambda_2\lambda_3 < 0$ from 3 negative λ's or from $(a+ib)(a-ib)\lambda_3 = (a^2+b^2)\lambda_3 < 0$.

If a real matrix has a complex eigenvalue $\lambda = a + ib$, then $\overline{\lambda} = a - ib$ is also an eigenvalue. The third eigenvalue must be real to make the trace real.

(b) The triangular matrix A has $\lambda = 1, 1, -5$ even with trace $= -3$ and $\det = -5$. There must be a third test for 3 by 3 matrices and that test must fail for this matrix.

25 You might think that $\boldsymbol{y}' = -A^2\boldsymbol{y}$ would always be stable because you are squaring the eigenvalues of A. But why is that equation unstable for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$?

This real matrix A has $\lambda = i$ and $-i$. Then $\lambda^2 = -1$ and -1. So $\boldsymbol{y}' = -A^2\boldsymbol{y}$ has eigenvalues 1 and 1 (unstable).

26 Find the three eigenvalues of A and the three roots of $s^3 - s^2 + s - 1 = 0$ (including $s = 1$). The equation $y''' - y'' + y' - y = 0$ becomes

$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} \quad \text{or} \quad \boldsymbol{z}' = A\boldsymbol{z}.$$

Each eigenvalue λ has an eigenvector $\boldsymbol{x} = (1, \lambda, \lambda^2)$.

$s^3 - s^2 + s - 1 = 0$ comes from substituting $y = e^{st}$ into $y''' - y'' + y' - y = 0$.

$\lambda^3 - \lambda^2 + \lambda - 1 = 0$ comes from computing $\det(A - \lambda I)$ for the 3 by 3 matrix.

One root is $s = 1$ (and $\lambda = 1$). The full cubic polynomial is

$s^3 - s^2 + s - 1 = (s-1)(s^2+1)$ with roots $\mathbf{1}, \boldsymbol{i}, -\boldsymbol{i}$.

Eigenvectors $(1, \lambda, \lambda^2) = (1,1,1), (1, i, -1), (1, -i, -1)$ for this companion matrix.

27 Find the two eigenvalues of A and the double root of $s^2 + 6s + 9 = 0$:

$$y'' + 6y' + 9y = 0 \text{ becomes } \begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} \text{ or } \boldsymbol{z}' = A\boldsymbol{z}.$$

The repeated eigenvalue gives only one solution $\boldsymbol{z} = e^{\lambda t}\boldsymbol{x}$. Find a second solution $\boldsymbol{z}$ from the second solution $y = te^{\lambda t}$.

The matrix has $\det(A - \lambda I) = \lambda^2 + 6\lambda + 9$. This is $\boldsymbol{(\lambda + 3)^2}$ so eigenvalues λ = roots $s = \mathbf{-3, -3}$. The two solutions are $y = e^{-3t}$ and $y = te^{-3t}$. Those translate to $\boldsymbol{z} = \begin{bmatrix} y \\ y' \end{bmatrix} = e^{-3t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\boldsymbol{z} = \begin{bmatrix} y \\ y' \end{bmatrix} = e^{-3t}\begin{bmatrix} t \\ 1 - 3t \end{bmatrix}$

28 Explain why a 3 by 3 companion matrix has eigenvectors $\boldsymbol{x} = (\mathbf{1}, \boldsymbol{\lambda}, \boldsymbol{\lambda^2})$.

First Way: If the first component is $x_1 = 1$, the first row of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ gives the second component $x_2 =$ _____. Then the second row of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ gives the third component $x_3 = \lambda^2$.

Second Way: $\boldsymbol{y}' = A\boldsymbol{y}$ starts with $y_1' = y_2$ and $y_2' = y_3$. $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ solves those equations. At $t = 0$ the equations become $\lambda x_1 = x_2$ and _____.

$A\boldsymbol{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -D & -C & -B \end{bmatrix}\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$ because rows 1 and 2 are true and row 3 is $-D - C\lambda - B\lambda^2 = \lambda^3$. That is $\lambda^3 + B\lambda^2 + C\lambda + D = 0$ corresponding to $y''' + By'' + Cy' + Dy = 0$.

29 Find A to change the scalar equation $y'' = 5y' - 4y$ into a vector equation for $\boldsymbol{z} = (y, y')$:

$$\frac{d\boldsymbol{z}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{z}.$$

What are the eigenvalues of the companion matrix A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' - 4y$.

$$\frac{d\boldsymbol{z}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ 5y' - 4y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 5 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{z}.$$

The eigenvalues come from $\lambda^2 - 5\lambda + 4 = 0$. Then $\lambda = \mathbf{1}$ and $\mathbf{4}$. Unstable because $y'' - 5y' + 4y$ has negative damping.

30 (a) Write down two familiar functions that solve the equation $d^2y/dt^2 = -9y$. Which one starts with $y(0) = 3$ and $y'(0) = 0$?

(b) This second-order equation $y'' = -9y$ produces a vector equation $\boldsymbol{z}' = A\boldsymbol{z}$:

$$\boldsymbol{z} = \begin{bmatrix} y \\ y' \end{bmatrix} \qquad \frac{d\boldsymbol{z}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{z}.$$

Find $\boldsymbol{z}(t)$ by using the eigenvalues and eigenvectors of A: $\boldsymbol{z}(0) = (3, 0)$.

(a) $y_1 = \cos 3t$ and $y_2 = \sin 3t$ and their combinations solve $y'' = -9y$. The initial conditions $y(0) = 3, y'(0) = 0$ are satisfied by $y = \mathbf{3\cos 3t}$.

(b) The matrix A has $\det \begin{bmatrix} -\lambda & 1 \\ -9 & -\lambda \end{bmatrix} = \lambda^2 + 9 = 0$ and $\lambda = \mathbf{3i}, \mathbf{-3i}$. Eigenvectors $(\mathbf{1}, \mathbf{3i}), (\mathbf{1}, \mathbf{-3i})$.

$\boldsymbol{z}(t) = c_1 e^{3it} \begin{bmatrix} 1 \\ 3i \end{bmatrix} + c_2 e^{-3it} \begin{bmatrix} 1 \\ -3i \end{bmatrix}$ gives $c_1 + c_2 = 3$ and $3ic_1 - 3ic_2 = 0$ at $t = 0$.

Then $c_1 = c_2 = \dfrac{3}{2}$ gives $\begin{bmatrix} y \\ y' \end{bmatrix} = \dfrac{3}{2} e^{3it} \begin{bmatrix} 1 \\ 3i \end{bmatrix} + \dfrac{3}{2} e^{-3it} \begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} \mathbf{3\cos 3t} \\ \mathbf{-9\sin 3t} \end{bmatrix}$.

31 (a) Change the third order equation $y''' - 2y'' - y' + 2y = 0$ to a first order system $\boldsymbol{z}' = A\boldsymbol{z}$ for the unknown $\boldsymbol{z} = (y, y', y'')$. The companion matrix A is 3 by 3.

(b) Substitute $y = e^{\lambda t}$ and also find $\det(A - \lambda I)$. Those lead to the same λ's.

(c) One root is $\lambda = 1$. Find the other roots and these complete solutions:

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t} \qquad \boldsymbol{z} = C_1 e^{\lambda_1 t} \boldsymbol{x}_1 + C_2 e^{\lambda_2 t} \boldsymbol{x}_2 + C_3 e^{\lambda_3 t} \boldsymbol{x}_3.$$

(a) $\boldsymbol{z}' = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} = A\boldsymbol{z}$

(b) The characteristic equation is $\det(A - \lambda I) = -(\lambda^3 - 2\lambda^2 - \lambda + 2) = 0$.

(c) $\lambda = 1$ is a root so we can factor out $(\lambda - 1)$:

$\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 1)(\lambda^2 - \lambda - 2) = (\lambda - 1)(\lambda - 2)(\lambda + 1)$ has roots $\mathbf{1}, \mathbf{2}, \mathbf{-1}$.

The complete solution is $y = c_1 e^t + c_2 e^{2t} + c_3 e^{-t}$.

This vectorizes into $\boldsymbol{z} = C_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

32 These companion matrices have $\lambda = 2, 1$ and $\lambda = 4, 1$. Find their eigenvectors:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ -4 & 5 \end{bmatrix} \qquad \text{Notice trace and determinant!}$$

A has $\lambda^2 - 3\lambda + 2 = 0 = (\lambda - 2)(\lambda - 1)$. $\lambda = \mathbf{2}, \mathbf{1}$ with eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

B has $\lambda^2 - 5\lambda + 4 = 0 = (\lambda - 4)(\lambda - 1)$. $\lambda = \mathbf{4}, \mathbf{1}$ with eigenvectors $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem Set 6.4, page 369

1 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, find an eigenvalue and an eigenvector of e^{At} and also of $-e^{-At}$.

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\boldsymbol{e^{At}x = e^{\lambda t}x}$ and $-e^{-At}\boldsymbol{x} = -e^{-\lambda t}\boldsymbol{x}$. Use the infinite series :

$$\begin{aligned} e^{At}\boldsymbol{x} &= (I + At + \tfrac{1}{2}(At)^2 + \cdots)\boldsymbol{x} \\ &= (I + \lambda t + \tfrac{1}{2}(\lambda t)^2 + \cdots)\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}. \end{aligned}$$

2 (a) From the infinite series $e^{At} = I + At + \cdots$ show that its derivative is Ae^{At}.

(b) The series for e^{At} ends quickly if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ because $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Find e^{At} and take its derivative (which should agree with Ae^{At}).

(a) The time derivative of the matrix e^{At} is Ae^{At} :

$\frac{d}{dt}(I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots) = A + A^2t + \frac{1}{2}A^3t^2 + \cdots) = Ae^{At}$.

(b) If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $A^2 = 0$ and $e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

The derivative of $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which agrees with Ae^{At}.

This derivative also agrees with A itself but that is an accident.

3 For $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ with eigenvectors in $V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, compute $e^{At} = Ve^{\Lambda t}V^{-1}$.

$$e^{At} = Ve^{\Lambda t}V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^t & \\ & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e^t} & \boldsymbol{e^{2t} - e^t} \\ \boldsymbol{0} & e^{2t} \end{bmatrix}.$$

Check $e^{At} = I$ at $t = 0$.

4 Why is $e^{(A+3I)t}$ equal to e^{At} multiplied by e^{3t} ?

If $AB = BA$ then $e^{(A+B)t} = e^{At}e^{Bt}$. (This usually fails if $AB \neq BA$.)

Here $B = 3I$ always gives $AB = BA$ so $e^{(A+3I)t} = e^{At}e^{3It} = e^{At}e^{3t}$ is **true**.

5 Why is $e^{A^{-1}}$ *not* the inverse of e^A ? What is the correct inverse of e^A ?

The correct inverse of e^A is e^{-A}. In general $e^{At}e^{AT} = e^{A(t+T)}$. Choose $t = 1, T = -1$.

The matrix $e^{A^{-1}}$ is a series of powers of A^{-1} and $(e^A)(e^{A^{-1}}) = e^{A+A^{-1}}$: not wanted.

6 Compute $A^n = \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}^n$. Add the series to find $e^{At} = \begin{bmatrix} e^t & c(e^t - 1) \\ 0 & 1 \end{bmatrix}$.

Start by assuming $\begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}^n = \begin{bmatrix} 1 & nc \\ 0 & 0 \end{bmatrix}$ (certainly true for ($n = 1$).

Then by induction $\begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & c \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & nc \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & (n+1)\,c \\ 0 & 0 \end{bmatrix}$.

The first equation is true for $n = 1$. Then the second equation says that every matrix multiplication adds c to the off-diagonal entry. So the first equation is true for $n = 2, 3, 4, \ldots$

Now add up the series for e^{At}:

$$I+At+\frac{1}{2}(At)^2+\cdots=\begin{bmatrix}1+t+\frac{1}{2}t^2+\cdots & 0+ct+\frac{1}{2}2ct^2+\cdots\\ 0 & 1+0+0+\cdots\end{bmatrix}=\begin{bmatrix}e^t & c(e^t-1)\\ 0 & 1\end{bmatrix}$$

7 Find e^A and e^B by using Problem 6 for $c=4$ and $c=-4$. Multiply to show that the matrices e^Ae^B and e^Be^A and e^{A+B} are all different.

$$A=\begin{bmatrix}1 & 4\\ 0 & 0\end{bmatrix}\qquad B=\begin{bmatrix}1 & -4\\ 0 & 0\end{bmatrix}\qquad A+B=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}.$$

With $t=1$ in Problem 6, $A=\begin{bmatrix}1 & 4\\ 0 & 0\end{bmatrix}$ has $e^A=\begin{bmatrix}e & 4(e-1)\\ 0 & 1\end{bmatrix}$

$B=\begin{bmatrix}1 & -4\\ 0 & 0\end{bmatrix}$ has $e^B=\begin{bmatrix}e & -4(e-1)\\ 0 & 1\end{bmatrix}$

Then $e^Ae^B=\begin{bmatrix}e^2 & (-4e+4)(e-1)\\ 0 & 1\end{bmatrix}$ and $e^Be^A=\begin{bmatrix}e^2 & (4e-4)(e-1)\\ 0 & 1\end{bmatrix}$ and $e^{A+B}=\begin{bmatrix}e^2 & 0\\ 0 & 1\end{bmatrix}$. Those three off-diagonal entries are different because AB and BA have off-diagonals -4 and 4.

8 Multiply the first terms $I+A+\frac{1}{2}A^2$ of e^A by the first terms $I+B+\frac{1}{2}B^2$ of e^B. Do you get the correct first three terms of e^{A+B}? *Conclusion*: e^{A+B} is not always equal to $(e^A)(e^B)$. The exponent rule only applies when $AB=BA$.

$\left(I+A+\frac{1}{2}A^2\right)\left(I+B+\frac{1}{2}B^2\right) = I+A+B+\frac{1}{2}A^2+AB+\frac{1}{2}B^2+\cdots$

The correct three terms of e^{A+B} are $I+A+B+\frac{1}{2}A^2+\frac{1}{2}AB+\frac{1}{2}BA+\frac{1}{2}B^2$.

Then AB agrees with $\frac{1}{2}AB+\frac{1}{2}BA$ **only if $\boldsymbol{AB=BA}$**.

9 Write $A=\left[\begin{smallmatrix}\mathbf{1} & \mathbf{4}\\ \mathbf{0} & \mathbf{0}\end{smallmatrix}\right]$ in the form $V\Lambda V^{-1}$. Find e^{At} from $Ve^{\Lambda t}V^{-1}$.

This is Problem 6 using diagonalization $A=V\Lambda V^{-1}$ by the eigenvector matrix V:

$$A=\begin{bmatrix}1 & 4\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & -4\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}$$

$$e^{At}=\begin{bmatrix}1 & -4\\ 0 & 1\end{bmatrix}\begin{bmatrix}e^t & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 4\\ 0 & 1\end{bmatrix}=\begin{bmatrix}e^t & 4(e^t-1)\\ 0 & 1\end{bmatrix}$$

10 Starting from $\boldsymbol{y}(0)$ the solution at time t is $e^{At}\boldsymbol{y}(0)$. Go an additional time t to reach $e^{At}\,e^{At}\boldsymbol{y}(0)$. Conclusion: e^{At} times e^{At} equals _____.

The conclusion is that e^{At} times e^{At} equals e^{2At}. No problem with $AB\neq BA$ because here B is the same as A.

11 Diagonalize A by V and confirm this formula for e^{At} by using $Ve^{\Lambda t}V^{-1}$:

$A=\begin{bmatrix}2 & 4\\ 0 & 3\end{bmatrix}\qquad e^{At}=\begin{bmatrix}e^{2t} & 4(e^{3t}-e^{2t})\\ 0 & e^{3t}\end{bmatrix}$ At $t=0$ this matrix is _____.

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = V\Lambda V^{-1}$$

$$e^{At} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & 4(e^{3t} - e^{2t}) \\ 0 & e^{3t} \end{bmatrix} = \boldsymbol{I} \textbf{ at } \boldsymbol{t = 0}.$$

12 (a) Find A^2 and A^3 and A^n for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ with repeated eigenvalues $\lambda = 1, 1$.

(b) Add the infinite series to find e^{At}. (The $Ve^{\Lambda t}V^{-1}$ method won't work.)

(a) $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. (b) $e^{At} =$

$$\begin{bmatrix} 1 + t + \frac{1}{2}t^2 + \cdots & t + \frac{1}{2}2t^2 + \frac{1}{6}3t^3 + \cdots \\ 0 & 1 + t + \frac{1}{2}t^2 + \cdots \end{bmatrix} = \begin{bmatrix} e^t & t(1 + t + \frac{1}{2}t^2 + \cdots) \\ 0 & e^t \end{bmatrix}$$

$$= \begin{bmatrix} e^t & \boldsymbol{te^t} \\ 0 & e^t \end{bmatrix}$$

Notice the factor t appearing as usual when there are equal roots (or equal eigenvalues).

13 (a) Solve $\boldsymbol{y}' = A\boldsymbol{y}$ as a combination of eigenvectors of this matrix A:

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{y} \quad \text{with } \boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(b) Write the equations as $y_1' = y_2$ and $y_2' = y_1$. Find an equation for y_1'' with y_2 eliminated. Solve for $y_1(t)$ and compare with part (a).

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $\lambda = 1$ with $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\lambda = -1$ with $x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Then $\boldsymbol{y}(0) = 4\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\boldsymbol{y}(t) = 4e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) If $y_1' = y_2$ and $y_2' = y_1$ then $y_1'' = y_2' = y_1$.

The second order equation $y_1'' = y_1$ has $y_1 = c_1e^t + c_2e^{-t}$.

The initial conditions produce the solution of part (a).

14 Similar matrices A and $B = V^{-1}AV$ have the *same eigenvalues* if V is invertible.

Second proof $\quad \det(V^{-1}AV - \lambda I) = (\det V^{-1})(\det(A - \lambda I))(\det V)$.

Why is this equation true? Then both sides are zero when $\det(A - \lambda I) = 0$.

We use the rule $\det ABC = (\det A)(\det B)(\det C)$.

Here $A = V^{-1}$ and $C = V$ have $(\det A)(\det C) = 1$.

This only leaves $\boldsymbol{\det B}$ which is $\boldsymbol{\det(A - \lambda I)}$.

Conclusion: $\boldsymbol{V^{-1}AV}$ **has the same eigenvalues as** $\boldsymbol{A}$. Similar matrices!

15 If B is *similar* to A, the growth rates for $\boldsymbol{z}' = B\boldsymbol{z}$ are the same as for $\boldsymbol{y}' = A\boldsymbol{y}$. That equation converts to the equation for $\boldsymbol{z}$ when $B = V^{-1}AV$ and $\boldsymbol{z} =$ ____.

If $\boldsymbol{y}' = A\boldsymbol{y}$ just set $\boldsymbol{y} = V\boldsymbol{z}$ to get $V\boldsymbol{z}' = AV\boldsymbol{z}$ which is $\boldsymbol{z}' = V^{-1}AV\boldsymbol{z}$. Similar matrices come from a change of variable in the differential equation.

16 If $A\boldsymbol{x} = \lambda\boldsymbol{x} \neq \boldsymbol{0}$, what is an eigenvalue and eigenvector of $(e^{At} - I)A^{-1}$?

The same $\boldsymbol{x}$ is an eigenvector, with eigenvalue in

$$(e^{At} - I)A^{-1}\boldsymbol{x} = \frac{1}{\lambda}(e^{At} - I)\boldsymbol{x} = \frac{e^{\lambda t} - 1}{\lambda}\boldsymbol{x}.$$

17 The matrix $B = \begin{bmatrix} \mathbf{0} & \mathbf{-4} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ has $B^2 = 0$. Find e^{Bt} from a (short) infinite series. Check that the derivative of e^{Bt} is Be^{Bt}.

$$e^{Bt} = I + Bt + 0 = \begin{bmatrix} 1 & -4t \\ 0 & 1 \end{bmatrix}. \text{ The derivative is } \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}.$$

The derivative is always Be^{Bt}; here it also equals B.

18 Starting from $\boldsymbol{y}(0) = \boldsymbol{0}$, solve $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q}$ as a combination of the eigenvectors. Suppose the source is $\boldsymbol{q} = q_1\boldsymbol{x}_1 + \cdots + q_n\boldsymbol{x}_n$. Solve for one eigenvector at a time, using the solution $y(t) = (e^{at} - 1)q/a$ to the scalar equation $y' = ay + q$.

Then $\boldsymbol{y}(t) = (e^{At} - I)A^{-1}\boldsymbol{q}$ is a combination of eigenvectors when all $\lambda_i \neq 0$.

For each eigenvector $\boldsymbol{x}$, a solution to $\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{x}$ is $\boldsymbol{y}(t) = \dfrac{e^{\lambda t} - 1}{\lambda}\boldsymbol{x}$ by Problem 16.

Then by linearity $\boldsymbol{y}(t) = \Sigma \dfrac{e^{\lambda_i t} - 1}{\lambda_i} q_i\boldsymbol{x}_i$ is the solution when $\boldsymbol{q} = q_1\boldsymbol{x}_1 + \cdots + q_n\boldsymbol{x}_n$. This is the same as $\boldsymbol{y}_p(t) = (e^{At} - I)A^{-1}\boldsymbol{q}$.

19 Solve for $\boldsymbol{y}(t)$ as a combination of the eigenvectors $\boldsymbol{x}_1 = (1,0)$ and $\boldsymbol{x}_2 = (1,1)$:

$$\boldsymbol{y}' = A\boldsymbol{y} + \boldsymbol{q} \qquad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \text{with} \quad \begin{matrix} y_1(0) = 0 \\ y_2(0) = 0 \end{matrix}$$

Write $\boldsymbol{q} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ as a combination $3\boldsymbol{x}_1 + \boldsymbol{x}_2$ of the eigenvectors of A. By Problem 18,

$$\boldsymbol{y}_p(t) = \frac{e^t - 1}{1}\, 3\boldsymbol{x}_1 + \frac{e^{2t} - 1}{2}\, \boldsymbol{x}_2.$$

20 Solve $\boldsymbol{y}' = A\boldsymbol{y} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}\boldsymbol{y}$ in three steps. First find the λ's and $\boldsymbol{x}$'s.

(1) Write $\boldsymbol{y}(0) = (3,1)$ as a combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$

(2) Multiply c_1 and c_2 by $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.

(3) Add the solutions $c_1e^{\lambda_1 t}\boldsymbol{x}_1 + c_2e^{\lambda_2 t}\boldsymbol{x}_2$.

Th eigenvalues come from $\det \begin{bmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1) = 0.$ Then $\lambda = \mathbf{4}$ and $\mathbf{-1}$.

The eigenvectors are found to be $\boldsymbol{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step (1) $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{4}{5}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step (2) Two solutions $\frac{4}{5}e^{4t}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\frac{3}{5}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Step (3) $\boldsymbol{y}(t) = \frac{4}{5}e^{4t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{3}{5}e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

21 Write five terms of the infinite series for e^{At}. Take the t derivative of each term. Show that you have four terms of Ae^{At}. Conclusion: $e^{At}\boldsymbol{y}(0)$ solves $d\boldsymbol{y}/dt = A\boldsymbol{y}$.

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \frac{1}{24}(At)^4 + \cdots$$
$$\frac{d}{dt}(e^{At} = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{4}A^4t^3 + \cdots = \boldsymbol{Ae^{At}}.$$

Problems 22-25 are about time-varying systems $\boldsymbol{y}' = \boldsymbol{A(t)y}$. Success then failure.

22 Suppose the constant matrix C has $C\boldsymbol{x} = \lambda\boldsymbol{x}$, and $p(t)$ is the integral of $a(t)$. Substitute $\boldsymbol{y} = e^{\lambda p(t)}\boldsymbol{x}$ to show that $d\boldsymbol{y}/dt = a(t)C\boldsymbol{y}$. Eigenvectors still solve this special time-varying system: constant matrix C multiplied by the scalar $a(t)$.

Here the time-varying coefficient matrix has the special form $a(t)C$, with the matrix C constant in time. Its eigenvalues and eigenvectors are $a(t)\lambda$ and $\boldsymbol{x}$ (main point: λ and $\boldsymbol{x}$ are constant). Then we can solve $\boldsymbol{y}' = a(t)C\boldsymbol{y}$ starting with an eigenvector:

$$\boldsymbol{y}(t) = e^{\int a(t)\lambda\, dt}\boldsymbol{x} \qquad \text{solves} \qquad \frac{d\boldsymbol{y}}{dt} = a(t)\lambda\boldsymbol{y} = a(t)C\boldsymbol{y}.$$

A combination of these solutions is also a solution—and can match $\boldsymbol{y}(0)$.

23 Continuing Problem 22, show from the series for $M(t) = e^{p(t)C}$ that $dM/dt = a(t)CM$. Then M is the fundamental matrix for the special system $\boldsymbol{y}' = a(t)C\boldsymbol{y}$. If $a(t) = 1$ then its integral is $p(t) = t$ and we recover $M = e^{Ct}$.

This question puts together the "fundamental matrix" $M(t)$ from Problem 22. Write $p(t) = \int a(t)\, dt$.

$$M = e^{p(t)C} = I + p(t)C + \frac{1}{2}p^2(t)C^2 + \cdots \qquad \text{and } \frac{dp}{dt} = a(t) \text{ give}$$
$$\frac{dM}{dt} = a(t)C + a(t)C^2p(t) + \cdots = \boldsymbol{a(t)CM}.$$

24 The integral of $A = \begin{bmatrix} 1 & 2t \\ 0 & 0 \end{bmatrix}$ is $P = \begin{bmatrix} t & t^2 \\ 0 & 0 \end{bmatrix}$. The exponential of P is $e^P = \begin{bmatrix} e^t & t(e^t-1) \\ 0 & 1 \end{bmatrix}$. From the chain rule we might hope that the derivative of

$e^{P(t)}$ is $P'e^{P(t)} = Ae^{P(t)}$. Compute the derivative of $e^{P(t)}$ and compare with the wrong answer $Ae^{P(t)}$. (One reason this feels wrong: Writing the chain rule as $(d/dt)e^P = e^P dP/dt$ would give $e^P A$ instead of $A\,e^P$. That is wrong too.)

Now the matrix $A(t)$ does not have the special form $A = a(t)C$ of problems 22–23. The problem shows that the simple formula doesn't solve $\boldsymbol{y}' = A(t)\boldsymbol{y}$. We can't just integrate $A(t)$ and use the matrix $e^{\int A(t)dt}$.

$$P = \int A(t)\,dt = \begin{bmatrix} t & t^2 \\ 0 & 0 \end{bmatrix} \quad \text{has} \quad P^2 = \begin{bmatrix} t^2 & t^3 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P^n = \begin{bmatrix} t^n & t^{n+1} \\ 0 & 0 \end{bmatrix}$$

Then $\dfrac{dP}{dt} = \begin{bmatrix} 1 & 2t \\ 0 & 0 \end{bmatrix} = A$ and $e^P = I + P + \frac{1}{2}P^2 + \cdots = \begin{bmatrix} e^t & te^t - t \\ 0 & 1 \end{bmatrix}$.

But the derivative of e^P is not $e^P \frac{dP}{dt}$. This matrix $e^{P(t)}$ is not solving $\boldsymbol{y}' = A(t)\boldsymbol{y}$.

25 Find the solution to $\boldsymbol{y}' = A(t)\boldsymbol{y}$ in Problem 24 by solving for y_2 and then y_1:

$$\text{Solve} \quad \begin{bmatrix} dy_1/dt \\ dy_2/dt \end{bmatrix} = \begin{bmatrix} 1 & 2t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ starting from } \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}.$$

Certainly $y_2(t)$ stays at $y_2(0)$. Find $y_1(t)$ by "undetermined coefficients" A, B, C: $y_1' = y_1 + 2ty_2(0)$ is solved by $y_1 = y_p + y_n = At + B + Ce^t$.

Choose A, B, C to satisfy the equation and match the initial condition $y_1(0)$.

The wrong answer in Problem 24 included the incorrect factor te^t in $e^{P(t)}$.

To solve $\boldsymbol{y}' = A(t)\boldsymbol{y}$ in Problem 24 we can start with its second equation:

$$\boldsymbol{y}' = A(t)\boldsymbol{y} \quad \text{is} \quad \begin{aligned} dy_1/dt &= y_1 + 2ty_2 \\ dy_2/dt &= 0 \end{aligned}$$

Then $y_2(t) = y_2(0) =$ constant and the first equation becomes $dy_1/dt = y_1 + 2ty_2(0)$. A particular solution has the form $y_1 = At + B$. Substitute this y_1 to find A and B:

$\dfrac{dy_1}{dt} = y_1 + 2ty_2(0)$ gives $A = At + B + 2ty_2(0)$ and then $A = -2y_2(0) = B$.

Now add a null solution Ce^t to start from $y_1(0)$:

$$y_1(t) = (y_1(0) + 2y_2(0))e^t - 2y_2(0)t - 2y_2(0).$$

This correct solution has no factor te^t.

Problem Set 6.5, page 379

Problems 1–14 are about eigenvalues. Then come differential equations.

1 Which of A, B, C have two real λ's? Which have two independent eigenvectors?

$$A = \begin{bmatrix} 7 & -11 \\ -11 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & -11 \\ 11 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 7 & -11 \\ 0 & 7 \end{bmatrix}$$

A is symmetric: Real λ's with a full set of two eigenvectors.

$B = 7I+$ antisymmetric: Complex $\lambda = 7 \pm 11i$, full set of (complex) eigenvectors.

$C = 7I - 11\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$: Eigenvalues $7, 7$ but only one eigenvector.

2 Show that A has real eigenvalues if $b \geq 0$ and nonreal eigenvalues if $b < 0$:

$$A = \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues of $\begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix}$ have $\lambda^2 - b = 0$. Then $\lambda = \pm\sqrt{b}$ if $b \geq 0$.

$\begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix}$ has $\lambda = \mathbf{1} \pm \sqrt{\boldsymbol{b}}$.

3 Find the eigenvalues and the unit eigenvectors of the symmetric matrices

$$\text{(a)}\ S = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{(b)}\ S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

(a) $\det \begin{bmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix} = (2-\lambda)\lambda^2 + 4\lambda + 4\lambda = -\lambda^3 + 2\lambda^2 + 8\lambda$

$= -\lambda(\lambda - 4)(\lambda + 2)$. $\lambda = \mathbf{0}, \mathbf{4}, \mathbf{-2}$.

Unit (orthonormal!) eigenvectors $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

(b) $\det \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{bmatrix} = \lambda(1-\lambda^2) + 4(1+\lambda) - 4(1-\lambda) = 9\lambda - \lambda^3$

$= -\lambda(\lambda - 3)(\lambda + 3)$.

$\lambda = \mathbf{0}, \mathbf{3}, \mathbf{-3}$ with orthonormal eigenvectors $\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

4 Find an orthogonal matrix Q that diagonalizes $S = \begin{bmatrix} \mathbf{-2} & \mathbf{6} \\ \mathbf{6} & \mathbf{7} \end{bmatrix}$. What is Λ?

The eigenvalues from $\lambda^2 - 5\lambda - 50 = 0 = (\lambda - 10)(\lambda + 5)$ are $\lambda_1 = \mathbf{10}$ and $\lambda_2 = \mathbf{5}$.
The unit eigenvectors are in Q:

$$Q = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad \text{with} \quad \Lambda = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}.$$

5 Show that this A (**symmetric but complex**) has only one line of eigenvectors:

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \text{ is not even diagonalizable. Its eigenvalues are } 0 \text{ and } 0.$$

$A^{\mathrm{T}} = A$ is not so special for complex matrices. *The good property is* $\overline{A}^{\mathrm{T}} = A$.
$\det(A - \lambda I) = \lambda^2$ gives $\lambda = \mathbf{0}, \mathbf{0}$. But $A - \lambda I = A$ has **rank 1**: Only one line of eigenvectors in its nullspace.

6 Find *all* orthogonal matrices from all $\boldsymbol{x}_1, \boldsymbol{x}_2$ to diagonalize $S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

$\lambda^2 - 25\lambda = 0$ gives eigenvalues $\mathbf{0}$ and $\mathbf{25}$. The (real) eigenvectors in Q can be

$$Q = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix} \text{ or } \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} \text{ or } \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix} \text{ or } \frac{1}{5}\begin{bmatrix} -4 & -3 \\ 3 & -4 \end{bmatrix}.$$

7 (a) Find a symmetric matrix $S = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.

(b) How do you know that S must have a negative pivot?

(c) How do you know that S can't have two negative eigenvalues?

The determinant of S is negative if $\boldsymbol{b^2 > 1}$. This determinant is (pivot 1)(pivot 2). Also $\det S = \lambda_1$ times λ_2. So exactly one eigenvalue is negative if $b^2 > 1$.

8 If $A^2 = 0$ then the eigenvalues of A must be ______. Give an example with $A \neq 0$. But if A is symmetric, diagonalize it to prove that the matrix is $A = 0$.

If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}$. Here $A^2 = 0$ so λ must be zero.

Nonsymmetric example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

The only symmetric example is $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ because $A = Q\Lambda Q^{\mathrm{T}}$ and $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

9 If $\lambda = a + ib$ is an eigenvalue of a real matrix A, then its conjugate $\overline{\lambda} = a - ib$ is also an eigenvalue. (If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then also $A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$.) Prove that every real 3 by 3 matrix has at least one real eigenvalue.

A real 3 by 3 matrix has $\det(A - \lambda I) = -\lambda^3 + c_2\lambda^2 + c_1\lambda + c_2 = 0$. If λ_1 satisfies this equation so does $\lambda_2 = \overline{\lambda_1}$ (take the conjugate of every term). But the sum $\lambda_1 + \lambda_2 + \lambda_3 =$ trace of $A =$ real number. So λ_3 must be real.

10 Here is a quick "proof" that the eigenvalues of *all* real matrices are real:

$$\textbf{False proof} \quad A\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{gives} \quad \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} = \lambda\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} \quad \text{so} \quad \lambda = \frac{\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}} \quad \text{is real.}$$

Find the flaw in this reasoning—a hidden assumption that is not justified. You could test those steps on the $90°$ rotation matrix $[\,0 \;\; -1; \;\; 1 \;\; 0\,]$ with $\lambda = i$ and $\boldsymbol{x} = (i, 1)$.

The flaw is to expect that $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}$ and $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ are real and $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} > 0$. When complex numbers are involved, it is $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{x}$ that is real and positive for every vector $\boldsymbol{x} \neq \mathbf{0}$.

11 Write A and B in the form $\lambda_1\boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \lambda_2\boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}}$ of the spectral theorem $Q\Lambda Q^T$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \qquad (\text{keep } \|\boldsymbol{x}_1\| = \|\boldsymbol{x}_2\| = 1).$$

A has $\lambda = \mathbf{4}, \mathbf{2}$ with unit eigenvectors in Q. Multiply columns times rows:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = Q\Lambda Q^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & \\ & 2 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= 4\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} + 2\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

B has $\lambda = 0, 25$ with these unit eigenvectors in Q:

$$\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}\begin{bmatrix} 0 & \\ & 25 \end{bmatrix}\begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix} = 0 + 25\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}\begin{bmatrix} 3/4 & 4/5 \end{bmatrix}.$$

12 What number b in $\begin{bmatrix} \mathbf{2} & \mathbf{b} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$ makes $A = Q\Lambda Q^{\mathrm{T}}$ possible? What number makes $A = V\Lambda V^{-1}$ impossible? What number makes A^{-1} impossible?

$b = \mathbf{1}$ makes A symmetric and then $A = Q\Lambda Q^{\mathrm{T}}$. $b = \mathbf{-1}$ makes $\lambda = 1,\ 1$ with only one eigenvector. $b = \mathbf{0}$ makes the matrix singular.

13 This A is nearly symmetric. But its eigenvectors are far from orthogonal:

$$A = \begin{bmatrix} 1 & 10^{-15} \\ 0 & 1 + 10^{-15} \end{bmatrix} \quad \text{has eigenvectors} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [\ ?\ ]$$

What is the dot product of the two unit eigenvectors ? A small angle !

The unit eigenvector for $\lambda = 1 + 10^{-15}$ is $\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The two eigenvectors are at a $45\,^\circ$ angle, far from orthogonal (even if A is nearly symmetric).

14 (Recommended) This matrix M is skew-symmetric and also orthogonal. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. They can only be i or $-i$. Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad \text{can only have eigenvalues } i \text{ or } -i.$$

The four eigenvalues must be $\lambda = \boldsymbol{i}, \boldsymbol{i}, \boldsymbol{-i}, \boldsymbol{-i}$ to produce trace = zero.

15 The complete solution to equation (8) for two oscillating springs (Figure 6.3) is

$$\boldsymbol{y}(t) = (A_1 \cos t + B_1 \sin t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (A_2 \cos\sqrt{3}t + B_2 \sin\sqrt{3}t)\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the numbers A_1, A_2, B_1, B_2 if $\boldsymbol{y}(0) = (3, 5)$ and $\boldsymbol{y}'(0) = (2, 0)$.

The numbers A_1, A_2 come from $y(0) = (3, 5)$ since $\cos 0 = 1$:

$$A_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \text{gives} \qquad A_1 = 4 \text{ and } A_2 = -1.$$

The numbers B_1, B_2 come from $y'(0) = (2,0)$ since $(\sin t)' = 1$ at $t = 0$ and $(\sin\sqrt{3}t)' = \sqrt{3}$ at $t = 0$:

$$B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sqrt{3}B_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{gives} \quad B_1 = B_2 = \frac{1}{\sqrt{3}}.$$

16 If the springs in Figure 6.3 have different constants k_1, k_2, k_3 then $\boldsymbol{y}'' + S\boldsymbol{y} = \boldsymbol{0}$ is

$$\begin{array}{ll} \text{Upper mass} & y_1'' + k_1 y_1 - k_2(y_2 - y_1) = 0 \\ \text{Lower mass} & y_2'' + k_2(y_2 - y_1) + k_3 y_2 = 0 \end{array} \qquad S = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

For $k_1 = 1, k_2 = 4, k_3 = 1$ find the eigenvalues $\lambda = \omega^2$ of S and the complete sine/cosine solution $\boldsymbol{y}(t)$ in equation (7).

The matrix $S = \begin{bmatrix} 1+4 & -4 \\ -4 & 4+1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1 = \omega_1^2$ and $\lambda_2 = 9 = \omega_2^2$. The complete solution to $\boldsymbol{y}'' + S\boldsymbol{y} = \boldsymbol{0}$ is

$$\boldsymbol{y}(t) = (A_1 \cos t + B_1 \sin t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (A_2 \cos 3t + B_2 \sin 3t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

17 Suppose the third spring is removed ($k_3 = 0$ and nothing is below mass 2). With $k_1 = 3, k_2 = 2$ in Problem 16, find S and its real eigenvalues and orthogonal eigenvectors. What is the sine/cosine solution $\boldsymbol{y}(t)$ if $\boldsymbol{y}(0) = (1,2)$ gives the cosines and $\boldsymbol{y}'(0) = (2,-1)$ gives the sines ?

When $k_1 = 3, k_2 = 2, k_3 = 0$, the matrix S becomes $S = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$ with $\lambda^2 - 7\lambda + 6 = (\lambda - 1)(\lambda - 6) = 0$.

The eigenvector for $\lambda_1 = \omega_1^2 = 1$ is $\boldsymbol{x}_1 = (1,2)$. The orthogonal eigenvector for $\lambda_2 = \omega_2^2 = 6$ is $\boldsymbol{x}_2 = (2,-1)$. Then $A_1 = 1$ and $A_2 = 0, B_1 = 0$ and $B_2 = 1/\sqrt{6}$ come from $\boldsymbol{y}(0) = \boldsymbol{x}_1$ and $\boldsymbol{y}'(0) = \boldsymbol{x}_2$. The solution to $\boldsymbol{y}'' + S\boldsymbol{y} = \boldsymbol{0}$ is $\boldsymbol{y}(t) = (\cos t)\boldsymbol{x}_1 + (\sin\sqrt{6}t)\boldsymbol{x}_2/\sqrt{6}$.

18 Suppose the top spring is also removed ($k_1 = 0$ and also $k_3 = 0$). S is singular! Find its eigenvalues and eigenvectors. If $\boldsymbol{y}(0) = (1,-1)$ and $\boldsymbol{y}' = (0,0)$ find $\boldsymbol{y}(t)$. If $\boldsymbol{y}(0)$ changes from $(1,-1)$ to $(1,1)$ what is $\boldsymbol{y}(t)$?

$S = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$ has $\lambda_1 = 0$ with $\boldsymbol{x}_1 = (1,1)$ and $\lambda_2 = 2k_2$ with $\boldsymbol{x}_2 = (1,-1)$.

$\boldsymbol{y}(0) = (1,-1)$ and $\boldsymbol{y}'(0) = (0,0)$ give $\boldsymbol{y}(t) = (\cos\sqrt{2k_2}\,t)\,\boldsymbol{x}_2$.

$\boldsymbol{y}(0) = (1, \ \ 1)$ and $\boldsymbol{y}'(0) = (0,0)$ give $\boldsymbol{y}(t) = \boldsymbol{x}_1 = (1,1)$: no movement!

There is no force from springs 1 and 3 and no initial velocity $\boldsymbol{y}'(0)$.

19 The matrix in this question is skew-symmetric ($A^{\mathrm{T}} = -A$). Energy is conserved.

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \boldsymbol{y} \quad \text{or} \quad \begin{aligned} y_1' &= cy_2 - by_3 \\ y_2' &= ay_3 - cy_1 \\ y_3' &= by_1 - ay_2. \end{aligned}$$

The derivative of $\|\boldsymbol{y}(t)\|^2 = y_1^2 + y_2^2 + y_3^2$ is $2y_1y_1' + 2y_2y_2' + 2y_3y_3'$. Substitute y_1', y_2', y_3' to get *zero*. The energy $\|\boldsymbol{y}(t)\|^2$ stays equal to $\|\boldsymbol{y}(0)\|^2$.

$y_1y_1' + y_2y_2' + y_3y_3' = y_1(cy_2 - by_3) + y_2(ay_3 - cy_1) + y_3(by_1 - ay_2) = \mathbf{0}$.

Then $\|\boldsymbol{y}(t)\|^2$ stays constant, equal to $\|\boldsymbol{y}(0)\|^2$.

20 *When $A = -A^{\mathrm{T}}$ is skew-symmetric,* e^{At} is ***orthogonal***. Prove $(e^{At})^{\mathrm{T}} = e^{-At}$ from the series $e^{At} = I + At + \frac{1}{2}A^2t^2 + \cdots$.

$A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$ has $\det = 9$: $\lambda = 3i$ and $-3i$ with $\boldsymbol{x} = (1, 3i)$ and $(1, -3i)$. Then $\boldsymbol{y}(t) = \frac{3}{2}e^{3it}\begin{bmatrix} 1 \\ 3i \end{bmatrix} + \frac{3}{2}e^{-3it}\begin{bmatrix} 1 \\ -3i \end{bmatrix} = \begin{bmatrix} 3\cos 3t \\ -9\sin 3t \end{bmatrix}$.

21 The mass matrix M can have masses $m_1 = 1$ and $m_2 = 2$. Show that the eigenvalues for $K\boldsymbol{x} = \lambda M\boldsymbol{x}$ are $\lambda = 2 \pm \sqrt{2}$, starting from $\det(K - \lambda M) = 0$:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } K = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \text{ are positive definite.}$$

Find the two eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Show that $\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2 \neq 0$ but $\boldsymbol{x}_1^{\mathrm{T}}M\boldsymbol{x}_2 = 0$.

$K\boldsymbol{x} = \lambda M\boldsymbol{x}$ is $(K - \lambda M)\boldsymbol{x} = \mathbf{0}$ and we need the determinant of $K - \lambda M$ to be 0:

$$\det\begin{bmatrix} 2-\lambda & -2 \\ -2 & 4-2\lambda \end{bmatrix} = 2(\lambda^2 - 4\lambda + 2) = 0 \qquad \lambda = \frac{4 \pm \sqrt{16-8}}{2} = \mathbf{2 \pm \sqrt{2}}.$$

The eigenvectors $\boldsymbol{x}_1 = (\mathbf{\sqrt{2}, -1})$ and $\boldsymbol{x}_2 = (\mathbf{\sqrt{2}, 1})$ come from

$(K - \lambda_1 M)\boldsymbol{x}_1 = \begin{bmatrix} -\sqrt{2} & -2 \\ -2 & -2\sqrt{2} \end{bmatrix}\boldsymbol{x}_1 = \mathbf{0}$ and $(K - \lambda_2 M)\boldsymbol{x}_2 = \begin{bmatrix} \sqrt{2} & -2 \\ -2 & 2\sqrt{2} \end{bmatrix}\boldsymbol{x}_2 = \mathbf{0}$.

Notice that $\boldsymbol{x}_1$ is **not** orthogonal to $\boldsymbol{x}_2$—it is "M-orthogonal":

$$\boldsymbol{x}_1^{\mathrm{T}}M\boldsymbol{x}_2 = \begin{bmatrix} \sqrt{2} & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} = 0.$$

22 What difference equation would you use to solve $\boldsymbol{y}'' = -S\boldsymbol{y}$?

$y'' = -Sy$ is well approximated by $y_{n+1} - 2y_n + y_{n-1} = -(\Delta t)^2 S y_n$. The initial conditions come in as $y_0 = y(0)$ and $y_1 = y(0) + \Delta t y'(0)$ (but that is only a first order accurate approximation to the true $y(\Delta t)$).

23 The second order equation $\boldsymbol{y}'' + S\boldsymbol{y} = \mathbf{0}$ reduces to a first order system $\boldsymbol{y}_1{}' = \boldsymbol{y}_2$ and $\boldsymbol{y}_2{}' = -S\boldsymbol{y}_1$. If $S\boldsymbol{x} = \omega^2\boldsymbol{x}$ show that the companion matrix $A = [0\ I\ ;\ -S\ 0]$ has eigenvalues $i\omega$ and $-i\omega$ with eigenvectors $(\boldsymbol{x}, i\omega\boldsymbol{x})$ and $(\boldsymbol{x}, -i\omega\boldsymbol{x})$.

The first-order equation with *block* companion matrix for $y'' = -Sy$ is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & I \\ -S & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & I \\ -S & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

For the eigenvalues: If $S\boldsymbol{x} = \omega^2\boldsymbol{x}$ then

$$\begin{bmatrix} 0 & I \\ -S & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{x} \\ \pm i\omega\boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \pm i\omega\boldsymbol{x} \\ -\omega^2\boldsymbol{x} \end{bmatrix} = \pm i\omega\begin{bmatrix} \boldsymbol{x} \\ \pm i\omega\boldsymbol{x} \end{bmatrix}.$$

So the block companion matrix A has eigenvalues $i\omega$ and $-i\omega$. Then we can compute and use the exponential e^{At} (if we want to).

24 Find the eigenvalues λ and eigenfunctions $y(x)$ for the differential equation $y'' = \lambda y$ with $y(0) = y(\pi) = 0$. There are infinitely many!

This is an important problem in function space—instead of eigenvectors in $\mathbf{R}^n$ we look for functions of x between $x = 0$ and $x = \pi$:

$$\frac{d^2y}{dt^2} = \lambda y(x) \text{ with boundary conditions } y(0) = y(\pi) = 0.$$

This equation is satisfied by $y(x) = a\cos\left(\sqrt{\lambda}\,x\right) + b\sin\left(\sqrt{\lambda}\,x\right)$.

The boundary condition $y(0) = 0$ makes $a = 0$.

The condition $y(\pi) = \sin\left(\sqrt{\lambda}\,\pi\right) = 0$ makes $\sqrt{\lambda} = \mathbf{1}$ or $\mathbf{2}$ or $\mathbf{3}$ or ... Then

$$\lambda = \mathbf{1^2} \text{ or } \mathbf{2^2} \text{ or any } \boldsymbol{n^2} \qquad y(x) = \sin(\sqrt{\lambda}\,x).$$