Preface

Differential equations and linear algebra are the two crucial courses in undergraduate mathematics. This new textbook develops those subjects separately and together. Separate is normal—these ideas are truly important. This book presents the basic course on differential equations, in full:

Chapter 1	First order equations
Chapter 2	Second order equations
Chapter 3	Graphical and numerical methods
Chapter 4	Matrices and linear systems
Chapter 6	Eigenvalues and eigenvectors

I will write below about the highlights and the support for readers. Here I focus on the option to include more linear algebra. Many colleges and universities want to move in this direction, by connecting two essential subjects.

More than ever, the central place of linear algebra is recognized. Limiting a student to the mechanics of matrix operations is over. Without planning it or foreseeing it, my lifework has been the presentation of linear algebra in books and video lectures:

Introduction to Linear Algebra (Wellesley–Cambridge Press)
MIT OpenCourseWare (**ocw.mit.edu**, Mathematics 18.06 in 2000 and 2014).

Linear algebra courses keep growing because the need keeps growing. At the same time, a rethinking of the MIT differential equations course 18.03 led to a new syllabus. And independently, it led to this book.

The underlying reason is that time is short and precious. The curriculum for many students is just about full. Still these two topics cannot be missed—and linear differential equations go in parallel with linear matrix equations. The prerequisite is calculus, for a single variable only—the key functions in these pages are inputs $f(t)$ and outputs $y(t)$. For all linear equations, continuous and discrete, the complete solution has two parts:

$$\textbf{One particular solution } \boldsymbol{y_p} \qquad \boldsymbol{A y_p = b}$$

$$\textbf{All null solutions } \boldsymbol{y_n} \qquad \boldsymbol{A y_n = 0}$$

Those right hand sides add to $\boldsymbol{b} + \boldsymbol{0} = \boldsymbol{b}$. The crucial point is that the left hand sides add to $\boldsymbol{A}(\boldsymbol{y_p} + \boldsymbol{y_n})$. When the inputs add, and the equation is linear, the outputs add. The equality $\boldsymbol{A}(\boldsymbol{y_p} + \boldsymbol{y_n}) = \boldsymbol{b} + \boldsymbol{0}$ tells us all solutions to $\boldsymbol{A y} = \boldsymbol{b}$:

$$\textbf{The complete solution to a linear equation is } \boldsymbol{y} = (\textbf{one } \boldsymbol{y_p}) + (\textbf{all } \boldsymbol{y_n}).$$

The same steps give the complete solution to $dy/dt = f(t)$, for the same reason. We know the answer from calculus—it is the form of the answer that is important here :

$$\frac{dy_p}{dt} = f(t) \quad \text{is solved by} \quad \boldsymbol{y_p(t) = \int_0^t f(x)\,dx}$$

$$\frac{dy_n}{dt} = 0 \quad \text{is solved by} \quad \boldsymbol{y_n(t) = C} \ \text{(any constant)}$$

$$\frac{dy}{dt} = f(t) \quad \text{is completely solved by} \quad \boldsymbol{y(t) = y_p(t) + C}$$

For every differential equation $dy/dt = Ay + f(t)$, our job is to find y_p and y_n : one particular solution and all homogeneous solutions. My deeper purpose is to build confidence, so the solution can be understood and used.

Differential Equations

The whole point of learning calculus is to understand movement. An economy grows, currents flow, the moon rises, messages travel, your hand moves. The action is fast or slow depending on forces from inside and outside : competition, pressure, voltage, desire. Calculus explains the meaning of dy/dt, but to stop without putting it into an equation (a differential equation) is to miss the whole purpose.

That equation may describe growth (often exponential growth e^{at}). It may describe oscillation and rotation (with sines and cosines). Very frequently the motion approaches an equilibrium, where forces balance. That balance point is found by linear algebra, when the rate of change dy/dt is zero.

The need is to explain what mathematics can do. I believe in looking partly outside mathematics, to include what scientists and engineers and economists actually remember and constantly use. My conclusion is that first place goes to linear equations. The essence of calculus is to linearize around a present position, to find the direction and the speed of movement.

Section 1.1 begins with the equations $dy/dt = y$ and $dy/dt = y^2$. It is simply wonderful that solving those two equations leads us here :

$$\boldsymbol{\frac{dy}{dt} = y \qquad y = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots \qquad y = e^t}$$

$$\boldsymbol{\frac{dy}{dt} = y^2 \qquad y = 1 + t + t^2 + t^3 + \cdots \qquad y = 1/(1-t)}$$

To meet the two most important series in mathematics, right at the start, that is pure pleasure. No better practice is possible as the course begins.

Important Choices of $f(t)$

Let me emphasize that a textbook must do more than solve random problems. We could invent functions $f(t)$ forever, but that is not right. Much better to understand a small number of highly important functions:

$f(t) =$	**sines and cosines**	(oscillating and rotating)
$f(t) =$	**exponentials**	(growing and decaying)
$f(t) =$	**1 for $t > 0$**	(a switch is turned on)
$f(t) =$	**impulse**	(a sudden shock)

The solution $y(t)$ is the response to those inputs—frequency response, exponential response, step response, impulse response. These particular functions and particular solutions are the best—the easiest to find and by far the most useful. All other solutions are built from these.

I know that an impulse (a delta function that acts in an instant) is new to most students. This idea deserves to be here! You will see how neatly it works. The response is like the inverse of a matrix—it gives a formula for *all* solutions. The book will be supplemented by video lectures on many topics like this, because a visual explanation can be so effective.

Support for Readers

Readers should know all the support that comes with this book:

math.mit.edu/dela is the key website. The time has passed for printing solutions to odd-numbered problems in the back of the book. The website can provide more detailed solutions and serious help. This includes additional worked problems, and codes for numerical experiments, and much more. Please make use of everything and contribute.

ocw.mit.edu has complete sets of video lectures on both subjects (OpenCourseWare is also on YouTube). Many students know about the linear algebra lectures for 18.06 and 18.06 SC. I am so happy they are helpful. For differential equations, the 18.03 SC videos and notes and exams are extremely useful.

The new videos will be about special topics—possibly even the Tumbling Box.

Linear Algebra

I must add more about linear algebra. My writing life has been an effort to present this subject clearly. Not abstractly, not with a minimum of words, but in a way that is helpful to the reader. It is such good fortune that the central ideas in matrix algebra (a basis for a vector space, factorization of matrices, the properties of symmetric and orthogonal matrices), are exactly the ideas that make this subject so useful. Chapter 5 emphasizes those ideas and Chapter 7 explains the applications of $A^{\mathrm{T}}A$.

Matrices are essential, not just optional. We are constantly acquiring and organizing and presenting data—the format we use most is a matrix. The goal is to see the relation between input and output. Often this relation is linear. In that case we can understand it.

The idea of a vector space is so central. Take *all* combinations of two vectors or two functions. I am always encouraging students to visualize that space—examples are really the best. When you see all solutions to $v_1 + v_2 + v_3 = 0$ and $d^2y/dt^2 + y = 0$, you have the idea of a vector space. This opens up the big questions of linear independence and basis and dimension—by example.

If $f(t)$ comes in continuous time, our model is a differential equation. If the input comes in discrete time steps, we use linear algebra. The model predicts the output $y(t)$ this is created by the input $f(t)$. But some inputs are simply more important than others—they are easier to understand and much more likely to appear. Those are the right equations to present in this course.

Notes to Faculty (and All Readers)

One reason for publishing with Wellesley-Cambridge Press can be mentioned here. I work hard to keep book costs reasonable for students. This was just as important for *Introduction to Linear Algebra*. A comparison on Amazon shows that textbook prices from big publishers are more than double. Wellesley-Cambridge books are distributed by SIAM inside North America and Cambridge University Press outside, and from Wellesley, with the same motive. Certainly quality comes first.

I hope you will see what this book offers. The first chapters are a normal textbook on differential equations, for a new generation. The complete book is a year's course on differential equations and linear algebra, including Fourier and Laplace transforms—plus PDE's (Laplace equation, heat equation, wave equation) and the FFT and the SVD.

This is extremely useful mathematics! I cannot hope that you will read every word. But why should the reader be asked to look elsewhere, when the applications can come so naturally here?

A special note goes to engineering faculty who look for support from mathematics. I have the good fortune to teach hundreds of engineering students every year. My work with finite elements and signal processing and computational science helped me to know what students need—and to speak their language. I see texts that mention the impulse response (for example) in one paragraph or not at all. But this is the fundamental solution from which all particular solutions come. In the book it is computed in the time domain, starting with e^{at}, and again with Laplace transforms. The website goes further.

I know from experience that every first edition needs help. I hope you will tell me what should be explained more clearly. You are holding a book with a valuable goal—to become a textbook for a world of students and readers in a new generation and a new time, with limits and pressing demands on that time. The book won't be perfect. I will be so grateful if you contribute, in any way, to making it better.

Acknowledgments

So many friends have helped this book. In first place is Ashley C. Fernandes, my early morning contact for 700 days. He leads the team at Valutone that prepared the LaTeX files. They gently allowed me to rewrite and rewrite, as the truly essential ideas of differential equations became clear. Working with friends is the happiest way to live.

The book began in discussions about the MIT course 18.03. Haynes Miller and David Jerison and Jerry Orloff wanted *change*—this is the lifeblood of a course. Think more about what we are doing ! Their starting point (I see it repeated all over the world) was to add more linear algebra. Matrix operations were already in 18.03, and computations of eigenvalues—they wanted bases and nullspaces and ideas.

I learned so much from their lectures. There is a wonderful moment when a class gets the point. Then the subject lives. The reader can feel this too, but only if the author does. I guess that is my philosophy of education.

Solutions to the Problem Sets were a gift from Bassel Khoury and Matt Ko. The example of a Tumbling Box came from Alar Toomre, it is the highlight of Section 3.3 (this was a famous experiment in his class, throwing a book in the air). Daniel Drucker watched over the text of Chapters 1-3, the best mathematics editor I know. My writing tries to be personal and direct—Dan tries to make it right.

The cover of this book was an amazing experience. Gonçalo Morais visited MIT from Portugal, and we talked. After he went home, he sent this very unusual picture of a strange attractor—a solution to the Lorenz equation. It became a way to honor that great and humble man, Ed Lorenz, who discovered chaos. Gail Corbett and Lois Sellers are the artists who created the cover—what they have done is beyond my thanks, it means everything.

At the last minute (every book has a crisis at the last minute) Shev MacNamara saved the day. Figures were missing. Big spaces were empty. The S-curve in Section 1.7, the direction fields in Section 3.1, the Euler and Runge-Kutta experiments, those and more came from Shev. He also encourages me to do an online course with new video lectures. I will think more about a MOOC when readers respond.

Thank you all, including every reader.

Gilbert Strang

Outline of Chapter 1 : First Order Equations

1.3	Solve	$\boldsymbol{dy/dt = ay}$	Construct the exponential e^{at}
1.4	Solve	$\boldsymbol{dy/dt = ay + q(t)}$	Four special $q(t)$ and all $q(t)$
1.5	Solve	$\boldsymbol{dy/dt = ay + e^{st}}$	Growth and oscillation : $s = a + i\omega$
1.6	Solve	$\boldsymbol{dy/dt = a(t)y + q(t)}$	Integrating factor $=$ 1/growth factor
1.7	Solve	$\boldsymbol{dy/dt = ay - by^2}$	The equation for $z = 1/y$ is linear
1.8	Solve	$\boldsymbol{dy/dt = g(t)/f(y)}$	Separate $\int f(y)\,dy$ from $\int g(t)\,dt$

The key formula in 1.4 gives the solution $\boldsymbol{y(t) = e^{at}y(0) + \int_0^t e^{a(t-s)}q(s)ds}$.

The website with solutions and codes and extra examples and videos is **math.mit.edu/dela**

Please contact **diffeqla@gmail.com** with questions and book orders and ideas.