Chapter 5

Vector Spaces and Subspaces

5.1 The Column Space of a Matrix

To a newcomer, matrix calculations involve a lot of numbers. To you, they involve vectors. The columns of $A\boldsymbol{v}$ and AB are linear combinations of n vectors—the columns of A. This chapter moves from numbers and vectors to a third level of understanding (the highest level). Instead of individual columns, we look at "spaces" of vectors. Without seeing ***vector spaces*** and their ***subspaces***, you haven't understood everything about $A\boldsymbol{v} = \boldsymbol{b}$.

Since this chapter goes a little deeper, it may seem a little harder. That is natural. We are looking inside the calculations, to find the mathematics. The author's job is to make it clear. Section 5.5 will present the "*Fundamental Theorem of Linear Algebra*."

We begin with the most important vector spaces. They are denoted by $\mathbf{R}^1$, $\mathbf{R}^2$, $\mathbf{R}^3$, $\mathbf{R}^4$, ... Each space $\mathbf{R}^n$ consists of a whole collection of vectors. $\mathbf{R}^5$ contains all column vectors with five components. This is called "5-dimensional space."

DEFINITION ***The space*** $\mathbf{R}^n$ ***consists of all column vectors*** v ***with n components.***

The components of $\boldsymbol{v}$ are real numbers, which is the reason for the letter $\mathbf{R}$. When the n components are complex numbers, $\boldsymbol{v}$ lies in the space $\mathbf{C}^n$.

The vector space $\mathbf{R}^2$ is represented by the usual xy plane. Each vector $\boldsymbol{v}$ in $\mathbf{R}^2$ has two components. The word "*space*" asks us to think of all those vectors—the whole plane. Each vector gives the x and y coordinates of a point in the plane: $\boldsymbol{v} = (x, y)$.

Similarly the vectors in $\mathbf{R}^3$ correspond to points (x, y, z) in three-dimensional space. The one-dimensional space $\mathbf{R}^1$ is a line (like the x axis). As before, we print vectors as a column between brackets, or along a line using commas and parentheses:

$$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \text{ is in } \mathbf{R}^2, \quad (1, 1, 0, 1, 1) \text{ is in } \mathbf{R}^5, \quad \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \text{ is in } \mathbf{C}^2.$$

The great thing about linear algebra is that it deals easily with five-dimensional space. We don't draw the vectors, we just need the five numbers (or n numbers).

To multiply $\boldsymbol{v}$ by 7, multiply every component by 7. Here 7 is a "scalar." To add vectors in $\mathbf{R}^5$, add them a component at a time : five additions. The two essential vector operations go on *inside the vector space*, and they produce ***linear combinations*** :

We can add any vectors in $\mathbf{R}^n$***, and we can multiply any vector*** v ***by any scalar*** c.

"Inside the vector space" means that ***the result stays in the space*** : This is crucial.

If $\boldsymbol{v}$ is in $\mathbf{R}^4$ with components $1, 0, 0, 1$, then $2\boldsymbol{v}$ is the vector in $\mathbf{R}^4$ with components $2, 0, 0, 2$. (In this case 2 is the scalar.) A whole series of properties can be verified in $\mathbf{R}^n$. The commutative law is $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$; the distributive law is $c(\boldsymbol{v} + \boldsymbol{w}) = c\boldsymbol{v} + c\boldsymbol{w}$. Every vector space has a unique "zero vector" satisfying $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$. Those are three of the eight conditions listed in the Chapter 5 Notes.

These eight conditions are required of every vector space. There are vectors other than column vectors, and there are vector spaces other than $\mathbf{R}^n$. All vector spaces have to obey the eight reasonable rules.

A real vector space is a set of "vectors" *together with rules for vector addition and multiplication by real numbers.* The addition and the multiplication must produce vectors that are in the space. And the eight conditions must be satisfied (which is usually no problem). You need to see three vector spaces other than $\mathbf{R}^n$:

M	The vector space of ***all real*** 2 ***by*** 2 ***matrices.***
Y	The vector space of ***all solutions*** $y(t)$ to $Ay'' + By' + Cy = 0$.
Z	The vector space that consists only of a ***zero vector***.

In **M** the "vectors" are really matrices. In **Y** the vectors are functions of t, like $y = e^{st}$. In **Z** the only addition is $\mathbf{0} + \mathbf{0} = \mathbf{0}$. In each space we can add : matrices to matrices, functions to functions, zero vector to zero vector. We can multiply a matrix by 4 or a function by 4 or the zero vector by 4. The result is still in **M** or **Y** or **Z**.

The space $\mathbf{R}^4$ is four-dimensional, and so is the space **M** of 2 by 2 matrices. Vectors in those spaces are determined by four numbers. The solution space **Y** is two-dimensional, because second order differential equations have two independent solutions. Section 5.4 will pin down those key words, *independence of vectors* and *dimension of a space*.

The space **Z** is zero-dimensional (by any reasonable definition of dimension). It is the smallest possible vector space. We hesitate to call it $\mathbf{R}^0$, which means no components—you might think there was no vector. *The vector space* **Z** *contains exactly one vector*. No space can do without that zero vector. Each space has its own zero vector—the zero matrix, the zero function, the vector $(0, 0, 0)$ in $\mathbf{R}^3$.

Subspaces

At different times, we will ask you to think of matrices and functions as vectors. But at all times, the vectors that we need most are ordinary column vectors. They are vectors with n components—but *maybe not all* of the vectors with n components. There are important vector spaces ***inside*** $\mathbf{R}^n$. Those are ***subspaces*** of $\mathbf{R}^n$.

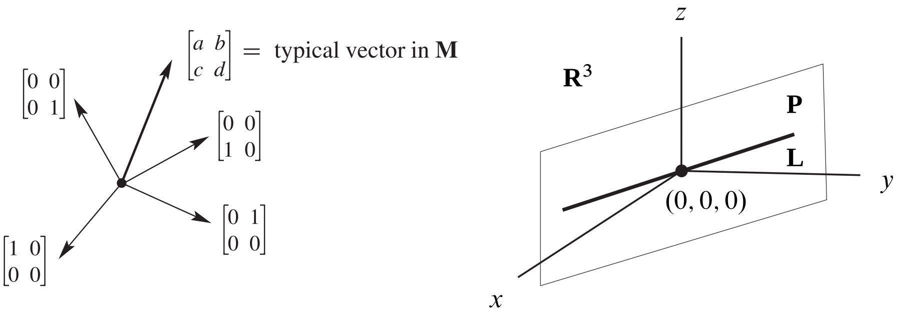


Figure 5.1: "4-dimensional" matrix space **M**. 3 subspaces of $\mathbf{R}^3$: plane **P**, line **L**, point **Z**.

Start with the usual three-dimensional space $\mathbf{R}^3$. Choose a plane through the origin $(0,0,0)$. ***That plane is a vector space in its own right.*** If we add two vectors in the plane, their sum is in the plane. If we multiply an in-plane vector by 2 or -5, it is still in the plane. A plane in three-dimensional space is not $\mathbf{R}^2$ (even if it looks like $\mathbf{R}^2$). The vectors have three components and they belong to $\mathbf{R}^3$. The plane **P** is a vector space ***inside*** $\mathbf{R}^3$.

This illustrates one of the most fundamental ideas in linear algebra. The plane going through $(0,0,0)$ is a ***subspace*** of the full vector space $\mathbf{R}^3$.

DEFINITION A ***subspace*** of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: ***If*** $\boldsymbol{v}$ ***and*** $\boldsymbol{w}$ ***are vectors in the subspace and*** c ***is any scalar, then***

(i) $\boldsymbol{v} + \boldsymbol{w}$ is in the subspace and **(ii)** $c\boldsymbol{v}$ is in the subspace.

In other words, the set of vectors is "closed" under addition $\boldsymbol{v} + \boldsymbol{w}$ and multiplication $c\boldsymbol{v}$ (and $d\boldsymbol{w}$). Those operations leave us in the subspace. We can also subtract, because $-\boldsymbol{w}$ is in the subspace and its sum with $\boldsymbol{v}$ is $\boldsymbol{v} - \boldsymbol{w}$. In short, ***all linear combinations*** $c\boldsymbol{v} + d\boldsymbol{w}$ ***stay in the subspace***.

First fact: ***Every subspace contains the zero vector***. The plane in $\mathbf{R}^3$ has to go through $(0,0,0)$. We mention this separately, for extra emphasis, but it follows directly from rule **(ii)**. Choose $c = 0$, and the rule requires $0\boldsymbol{v}$ to be in the subspace.

Planes that don't contain the origin fail those tests. When $\boldsymbol{v}$ is on such a plane, $-\boldsymbol{v}$ and $0\boldsymbol{v}$ are *not* on the plane. A plane that misses the origin is not a subspace.

Lines through the origin are also subspaces. When we multiply by 5, or add two vectors on the line, we stay on the line. But the line must go through $(0,0,0)$.

Another subspace is all of $\mathbf{R}^3$. The whole space is a subspace (*of itself*). That is a fourth subspace in the figure. Here is a list of all the possible subspaces of $\mathbf{R}^3$:

(**L**) Any line through $(0,0,0)$ ($\mathbf{R}^3$) The whole space
(**P**) Any plane through $(0,0,0)$ (**Z**) The single vector $(0,0,0)$

If we try to keep only *part* of a plane or line, the requirements for a subspace don't hold. Look at these examples in $\mathbf{R}^2$.

Example 1 Keep only the vectors (x, y) whose components are positive or zero (this is a quarter-plane). The vector $(2, 3)$ is included but $(-2, -3)$ is not. So rule **(ii)** is violated when we try to multiply by $c = -1$. ***The quarter-plane is not a subspace***.

Example 2 Include also the vectors whose components are both negative. Now we have two quarter-planes. Requirement **(ii)** is satisfied; we can multiply by any c. But rule **(i)** now fails. The sum of $\boldsymbol{v} = (2, 3)$ and $\boldsymbol{w} = (-3, -2)$ is $(-1, 1)$, which is outside the quarter-planes. ***Two quarter-planes don't make a subspace***.

Rules **(i)** and **(ii)** involve vector addition $\boldsymbol{v} + \boldsymbol{w}$ and multiplication by scalars like c and d. The rules can be combined into a single requirement—*the rule for subspaces*:

A subspace containing v and w must contain all linear combinations $cv + d\boldsymbol{w}$.

Example 3 Inside the vector space $\mathbf{M}$ of all 2 by 2 matrices, here are two subspaces:

$$(\mathbf{U}) \text{ All upper triangular matrices } \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \qquad (\mathbf{D}) \text{ All diagonal matrices } \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Add any two matrices in $\mathbf{U}$, and the sum is in $\mathbf{U}$. Add diagonal matrices, and the sum is diagonal. In this case $\mathbf{D}$ is also a subspace of $\mathbf{U}$! The zero matrix alone is also a subspace, when a, b, and d all equal zero.

For a smaller subspace of diagonal matrices, we could require $a = d$. The matrices are multiples of the identity matrix I. These aI form a "line of matrices" in $\mathbf{M}$ and $\mathbf{U}$ and $\mathbf{D}$.

Is the matrix I a subspace by itself? Certainly not. Only the zero matrix is. Your mind will invent more subspaces of 2 by 2 matrices—write them down for Problem 6.

The Column Space of A

The most important subspaces are tied directly to a matrix A. We are trying to solve $A\boldsymbol{v} = \boldsymbol{b}$. If A is not invertible, the system is solvable for some $\boldsymbol{b}$ and not solvable for other $\boldsymbol{b}$. We want to describe the good right sides $\boldsymbol{b}$—the vectors that *can* be written as A times $\boldsymbol{v}$. Those $\boldsymbol{b}'s$ form the "*column space*" of A.

Remember that $A\boldsymbol{v}$ is a combination of the columns of A. To get every possible $\boldsymbol{b}$, we use every possible $\boldsymbol{v}$. Start with the columns of A, and ***take all their linear combinations. This produces the column space of A.*** It contains not just the n columns of A!

DEFINITION ***The column space consists of all combinations of the columns***.

The combinations are all possible vectors $A\boldsymbol{v}$. They fill the column space $\boldsymbol{C}(A)$.

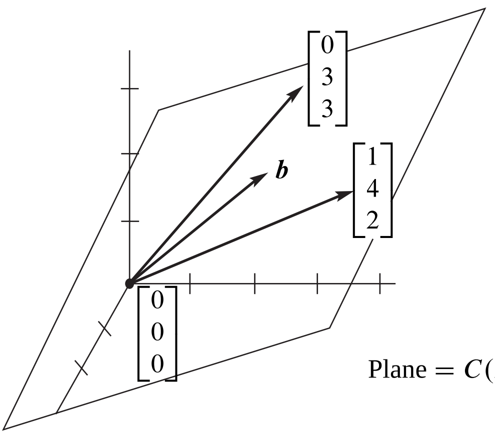
This column space is crucial to the whole book, and here is why. ***To solve $A\boldsymbol{v} = \boldsymbol{b}$ is to express $\boldsymbol{b}$ as a combination of the columns. The right side $\boldsymbol{b}$ has to be in the column space*** produced by A on the left side. If $\boldsymbol{b}$ is not in $\boldsymbol{C}(A)$, $A\boldsymbol{v} = \boldsymbol{b}$ has no solution.

The system $Av = b$ is solvable if and only if b is in the column space of A.

When $\boldsymbol{b}$ is in the column space, it is a combination of the columns. The coefficients in that combination give us a solution $\boldsymbol{v}$ to the system $A\boldsymbol{v} = \boldsymbol{b}$.

Suppose A is an m by n matrix. Its columns have m components (not n). So the columns belong to $\mathbf{R}^m$. ***The column space of A is a subspace of $\mathbf{R}^m$ (not $\mathbf{R}^n$).*** The set of all column combinations $A\boldsymbol{x}$ satisfies rules (**i**) and (**ii**) for a subspace: When we add linear combinations or multiply by scalars, we still produce combinations of the columns. The word "subspace" is always justified *by taking all linear combinations*.

Here is a 3 by 2 matrix A, whose column space is a subspace of $\mathbf{R}^3$. The column space of A is a plane in Figure 5.2.



$$A = \begin{bmatrix}1 & 0\\4 & 3\\2 & 3\end{bmatrix}$$

$$\boldsymbol{b} = v_1\begin{bmatrix}1\\4\\2\end{bmatrix} + v_2\begin{bmatrix}0\\3\\3\end{bmatrix}$$

Plane $= C(A) =$ all vectors $A\boldsymbol{v}$

Figure 5.2: The column space $\boldsymbol{C}(A)$ is a plane containing the two columns of A. $A\boldsymbol{v} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is on that plane. Then $\boldsymbol{b}$ is a combination of the columns.

We drew one particular $\boldsymbol{b}$ (a combination of the columns). This $\boldsymbol{b} = A\boldsymbol{v}$ lies on the plane. The plane has zero thickness, so most right sides $\boldsymbol{b}$ in $\mathbf{R}^3$ are *not* in the column space. For most $\boldsymbol{b}$ there is no solution to our 3 equations in 2 unknowns.

Of course $(0, 0, 0)$ is in the column space. The plane passes through the origin. There is certainly a solution to $A\boldsymbol{v} = \mathbf{0}$. That solution, always available, is $\boldsymbol{v} =$ ______.

To repeat, the attainable right sides $\boldsymbol{b}$ are exactly the vectors in the column space. One possibility is the first column itself—take $v_1 = 1$ and $v_2 = 0$. Another combination is the second column—take $v_1 = 0$ and $v_2 = 1$. The new level of understanding is to see *all* combinations—the whole subspace is generated by those two columns.

Notation The column space of A is denoted by $\boldsymbol{C}(A)$. Start with the columns and take all their linear combinations. We might get the whole $\mathbf{R}^m$ or only a small subspace.

Important Instead of columns in $\mathbf{R}^m$, we could start with any set of vectors in a vector space $\mathbf{V}$. To get a subspace $\mathbf{SS}$ of $\mathbf{V}$, we take *all combinations* of the vectors in that set:

$$\begin{aligned} \mathbf{S} &= \text{set of vectors } \boldsymbol{s} \text{ in } \mathbf{V} \ (\mathbf{S} \text{ is probably } \textit{not} \text{ a subspace}) \\ \mathbf{SS} &= \text{all combinations of vectors in } \mathbf{S} \ (\mathbf{SS} \textit{ is} \text{ a subspace}) \end{aligned}$$

$$\mathbf{SS} = \boxed{\text{all } c_1\boldsymbol{s}_1 + \cdots + c_N\boldsymbol{s}_N} = \boxed{\textbf{the subspace of V} \text{ "spanned" by } \mathbf{S}}$$

When $\mathbf{S}$ is the set of columns, $\mathbf{SS}$ is the column space. When there is only one nonzero vector $\boldsymbol{v}$ in $\mathbf{S}$, the subspace $\mathbf{SS}$ is the line through $\boldsymbol{v}$. *Always* $\mathbf{SS}$ *is the smallest subspace containing* $\mathbf{S}$. This is a fundamental way to create subspaces and we will come back to it.

The subspace SS is the "span" of S, containing all combinations of vectors in S.

Example 4 Describe the column spaces (they are subspaces of $\mathbf{R}^2$) for these matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution The column space of I is the *whole space* $\mathbf{R}^2$. Every vector is a combination of the columns of I. In vector space language, $\boldsymbol{C}(I)$ equals $\mathbf{R}^2$.

The column space of A is only a line. The second column $(2,4)$ is a multiple of the first column $(1,2)$. Those vectors are different, but our eye is on vector ***spaces***. The column space contains $(1,2)$ and $(2,4)$ and all other vectors $(c,2c)$ along that line. The equation $A\boldsymbol{v} = \boldsymbol{b}$ is only solvable when $\boldsymbol{b}$ is on the line.

For the third matrix (with three columns) the column space $\boldsymbol{C}(B)$ is all of $\mathbf{R}^2$. Every $\boldsymbol{b}$ is attainable. The vector $\boldsymbol{b} = (5,4)$ is column 2 plus column 3, so $\boldsymbol{v}$ can be $(0,1,1)$. The same vector $(5,4)$ is also 2(column 1) + column 3, so another possible $\boldsymbol{v}$ is $(2,0,1)$. This matrix has the same column space as I—any $\boldsymbol{b}$ is allowed. But now $\boldsymbol{v}$ has extra components and $A\boldsymbol{v} = \boldsymbol{b}$ has more solutions—more combinations that give $\boldsymbol{b}$.

The next section creates the *nullspace* $\boldsymbol{N}(A)$, to describe all the solutions of $A\boldsymbol{v} = \mathbf{0}$. This section created the column space $\boldsymbol{C}(A)$, to describe all the attainable right sides $\boldsymbol{b}$.

■ REVIEW OF THE KEY IDEAS ■

1. $\mathbf{R}^n$ contains all column vectors with n real components.

2. $\mathbf{M}$ (2 by 2 matrices) and $\mathbf{Y}$ (functions) and $\mathbf{Z}$ (zero vector alone) are vector spaces.

3. A subspace containing $\boldsymbol{v}$ and $\boldsymbol{w}$ must contain all their combinations $c\boldsymbol{v} + d\boldsymbol{w}$.

4. The combinations of the columns of A form the ***column space*** $\boldsymbol{C}(A)$. Then the column space is "spanned" by the columns.

5. $A\boldsymbol{v} = \boldsymbol{b}$ has a solution exactly when $\boldsymbol{b}$ is in the column space of A.

■ WORKED EXAMPLES ■

5.1 A We are given three different vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$. Construct a matrix so that the equations $A\boldsymbol{v} = \boldsymbol{b}_1$ and $A\boldsymbol{v} = \boldsymbol{b}_2$ are solvable, but $A\boldsymbol{v} = \boldsymbol{b}_3$ is *not* solvable. How can you decide if this is possible? How could you construct A?

Solution We want to have $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ in the column space of A. Then $A\boldsymbol{v} = \boldsymbol{b}_1$ and $A\boldsymbol{v} = \boldsymbol{b}_2$ will be solvable. *The quickest way is to make $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ the two columns of A.* Then the solutions are $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{v} = (0, 1)$.

Also, we don't want $A\boldsymbol{v} = \boldsymbol{b}_3$ to be solvable. So don't make the column space any larger! Keeping only the columns $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, the question is: *Do we already have $\boldsymbol{b}_3$?*

$$\text{Is } A\boldsymbol{v} = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \boldsymbol{b}_3 \text{ solvable?} \qquad \text{Is } \boldsymbol{b}_3 \text{ a combination of } \boldsymbol{b}_1 \text{ and } \boldsymbol{b}_2\text{?}$$

If the answer is *no*, we have the desired matrix A. If $\boldsymbol{b}_3$ *is* a combination of $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, then it is *not possible* to construct A. The column space $\boldsymbol{C}(A)$ will have to contain $\boldsymbol{b}_3$.

5.1 B Describe a subspace **S** of each vector space **V**, and then a subspace **SS** of **S**.

$$\begin{aligned} \mathbf{V}_3 &= \text{all combinations of } (1,1,0,0) \text{ and } (1,1,1,0) \text{ and } (1,1,1,1) \\ \mathbf{V}_2 &= \text{all vectors } \boldsymbol{v} \text{ perpendicular to } \boldsymbol{u} = (1,2,1), \text{ so } \boldsymbol{u}\cdot\boldsymbol{v} = 0 \\ \mathbf{V}_4 &= \text{all solutions } y(x) \text{ to the equation } d^4y/dx^4 = 0 \end{aligned}$$

Describe each **V** two ways: (1) *All combinations of* .... (2) *All solutions of* ....

Solution $\mathbf{V}_3$ starts with three vectors. A subspace **S** comes from all combinations of the first two vectors $(1, 1, 0, 0)$ and $(1, 1, 1, 0)$. A subspace **SS** of **S** comes from all multiples $(c, c, 0, 0)$ of the first vector. So many possibilities.

A subspace **S** of $\mathbf{V}_2$ is the line through $(1, -1, 1)$. This line is perpendicular to $\boldsymbol{u}$. The zero vector $\boldsymbol{z} = (0, 0, 0)$ is in **S**. The smallest subspace **SS** is **Z**.

$\mathbf{V}_4$ contains all cubic polynomials $y = a + bx + cx^2 + dx^3$, with $d^4y/dx^4 = 0$. The quadratic polynomials (without an x^3 term) give a subspace **S**. The linear polynomials are one choice of **SS**. The constants $y = a$ could be **SSS**.

In all three parts we could take **S** = **V** itself, and **SS** = the zero subspace **Z**.

Each **V** can be described as *all combinations of* .... and as *all solutions of* ....:

$$\begin{aligned} \mathbf{V}_3 &= \text{all combinations of the 3 vectors} & \mathbf{V}_3 &= \text{all solutions of } v_1 - v_2 = 0. \\ \mathbf{V}_2 &= \text{all combinations of } (1,0,-1) \text{ and } (1,-1,1) & \mathbf{V}_2 &= \text{all solutions of } \boldsymbol{u}\cdot\boldsymbol{v} = 0. \\ \mathbf{V}_4 &= \text{all combinations of } 1, x, x^2, x^3 & \mathbf{V}_4 &= \text{all solutions to } d^4y/dx^4 = 0. \end{aligned}$$

Problem Set 5.1

Questions 1–10 are about the "subspace requirements": $\boldsymbol{v}+\boldsymbol{w}$ and $c\boldsymbol{v}$ (and then all linear combinations $c\boldsymbol{v}+d\boldsymbol{w}$) stay in the subspace.

1 One requirement can be met while the other fails. Show this by finding

(a) A set of vectors in $\mathbf{R}^2$ for which $\boldsymbol{v}+\boldsymbol{w}$ stays in the set but $\frac{1}{2}\boldsymbol{v}$ may be outside.

(b) A set of vectors in $\mathbf{R}^2$ (other than two quarter-planes) for which every $c\boldsymbol{v}$ stays in the set but $\boldsymbol{v}+\boldsymbol{w}$ may be outside.

2 Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\boldsymbol{v} = (1,4,0)$ and $\boldsymbol{w} = (2,2,2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \le b_2 \le b_3$.

3 Describe the smallest subspace of the matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

4 Let $\mathbf{P}$ be the plane in $\mathbf{R}^3$ with equation $x + y - 2z = 4$. The origin $(0,0,0)$ is not in $\mathbf{P}$! Find two vectors in $\mathbf{P}$ and check that their sum is not in $\mathbf{P}$.

5 Let $\mathbf{P}_0$ be the plane through $(0,0,0)$ parallel to the previous plane $\mathbf{P}$. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

6 The subspaces of $\mathbf{R}^3$ are planes, lines, $\mathbf{R}^3$ itself, or $\mathbf{Z}$ containing only $(0,0,0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe all subspaces of $\mathbf{D}$, the space of 2 by 2 diagonal matrices.

7 (a) The intersection of two planes through $(0,0,0)$ is probably a ______ but it could be a ______. It can't be $\mathbf{Z}$!

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a ______ but it could be a ______.

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, prove that their intersection $\mathbf{S} \cap \mathbf{T}$ is a subspace of $\mathbf{R}^5$. Here $\mathbf{S} \cap \mathbf{T}$ consists of the vectors that lie in both subspaces. *Check the requirements on $\boldsymbol{v}+\boldsymbol{w}$ and $c\boldsymbol{v}$.*

8 Suppose $\mathbf{P}$ is a plane through $(0,0,0)$ and $\mathbf{L}$ is a line through $(0,0,0)$. The smallest vector space $\mathbf{P}+\mathbf{L}$ containing both $\mathbf{P}$ and $\mathbf{L}$ is either ______ or ______.

9 (a) Show that the set of *invertible* matrices in **M** is not a subspace.

(b) Show that the set of *singular* matrices in **M** is not a subspace.

10 True or false (check addition in each case by an example):

(a) The symmetric matrices in **M** (with $A^{\mathrm{T}} = A$) form a subspace.

(b) The skew-symmetric matrices in **M** (with $A^{\mathrm{T}} = -A$) form a subspace.

(c) The unsymmetric matrices in **M** (with $A^{\mathrm{T}} \neq A$) form a subspace.

Questions 11–19 are about column spaces $C(A)$ and the equation $A\boldsymbol{v} = \boldsymbol{b}$.

11 Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

12 For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

13 Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. Which matrices have the same column space? Which have the same *row space*?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

14 For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

15 (Recommended) If we add an extra column $\boldsymbol{b}$ to a matrix A, then the column space gets larger unless ______. Give an example where the column space gets larger and an example where it doesn't. Why is $A\boldsymbol{v} = \boldsymbol{b}$ solvable exactly when the column space *doesn't* get larger? Then it is the same for A and $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$.

16 The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A.* Give an example where the column spaces of A and AB are not equal.

17 Suppose $A\boldsymbol{v} = \boldsymbol{b}$ and $A\boldsymbol{w} = \boldsymbol{b}^*$ are both solvable. Then $A\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{b}^*$ is solvable. What is $\boldsymbol{z}$? This translates into: If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in the column space $\boldsymbol{C}(A)$, then $\boldsymbol{b} + \boldsymbol{b}^*$ is also in $\boldsymbol{C}(A)$.

18 If A is any 5 by 5 invertible matrix, then its column space is ______. Why?

19 True or false (with a counterexample if false):

(a) The vectors $\boldsymbol{b}$ that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A (test this).

20 Construct a 3 by 3 matrix whose column space contains $(1, 1, 0)$ and $(1, 0, 1)$ but not $(1, 1, 1)$. Construct a 3 by 3 matrix whose column space is only a line.

21 If the 9 by 12 system $A\boldsymbol{v} = \boldsymbol{b}$ is solvable for every $\boldsymbol{b}$, then $\boldsymbol{C}(A)$ must be ______.

Challenge Problems

22 Suppose $\mathbf{S}$ and $\mathbf{T}$ are two subspaces of a vector space $\mathbf{V}$. The **sum** $\mathbf{S} + \mathbf{T}$ contains all sums $\boldsymbol{s} + \boldsymbol{t}$ of a vector $\boldsymbol{s}$ in $\mathbf{S}$ and a vector $\boldsymbol{t}$ in $\mathbf{T}$. Then $\mathbf{S} + \mathbf{T}$ is a vector space.

If $\mathbf{S}$ and $\mathbf{T}$ are lines in $\mathbf{R}^m$, what is the difference between $\mathbf{S} + \mathbf{T}$ and $\mathbf{S} \cup \mathbf{T}$? That union contains all vectors from $\mathbf{S}$ and all vectors from $\mathbf{T}$. Explain this statement: *The span of* $\mathbf{S} \cup \mathbf{T}$ *is* $\mathbf{S} + \mathbf{T}$.

23 If $\mathbf{S}$ is the column space of A and $\mathbf{T}$ is $\boldsymbol{C}(B)$, then $\mathbf{S} + \mathbf{T}$ is the column space of what matrix M? The columns of A and B and M are all in $\mathbf{R}^m$. (I don't think $A + B$ is always a correct M.)

24 Show that the matrices A and $\begin{bmatrix} A & AB \end{bmatrix}$ (this has extra columns) have the same column space. But find a square matrix with $\boldsymbol{C}(A^2)$ smaller than $\boldsymbol{C}(A)$.

25 An n by n matrix has $\boldsymbol{C}(A) = \mathbf{R}^n$ exactly when A is an ______ matrix.