

Chapter 4

Linear Equations and Inverse Matrices

4.1 Two Pictures of Linear Equations

The central problem of linear algebra is to solve a system of equations. Those equations are linear, which means that the unknowns are only multiplied by numbers—we never see x^2 or x times y . Our first linear system is deceptively small, only “2 by 2.” But you will see how far it leads :

Two equations	$x - 2y = 1$	(1)
Two unknowns	$2x + y = 7$	

We begin *a row at a time*. The first equation $x - 2y = 1$ produces a straight line in the xy plane. The point $x = 1, y = 0$ is on the line because it solves that equation. The point $x = 3, y = 1$ is also on the line because $3 - 2 = 1$. For $x = 101$ we find $y = 50$.

The slope of this line in Figure 4.1 is $\frac{1}{2}$, because y increases by 1 when x changes by 2. But slopes are important in calculus and this is linear algebra !

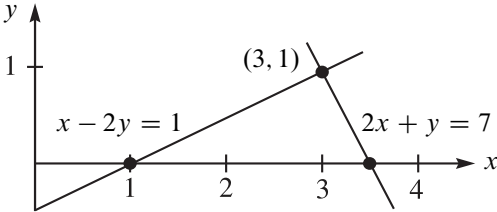


Figure 4.1: Row picture : The point (3, 1) where the two lines meet is the solution.

The second line in this “row picture” comes from the second equation $2x + y = 7$. You can’t miss the intersection point where the two lines meet. *The point $x = 3, y = 1$ lies on both lines.* It solves both equations at once. This is the solution to our two equations.

ROWS *The row picture shows two lines meeting at a single point (the solution).*

Turn now to the column picture. I want to recognize the same linear system as a “vector equation.” Instead of numbers we need to see *vectors*. If you separate the original system into its columns instead of its rows, you get a vector equation :

Combination equals b
$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} = b. \quad (2)$$

This has two column vectors on the left side. The problem is *to find the combination of those vectors that equals the vector on the right.* We are multiplying the first column by x and the second column by y , and adding vectors. With the right choices $x = 3$ and $y = 1$ (the same numbers as before), this produces $3(\text{column 1}) + 1(\text{column 2}) = b$.

COLUMNS *The column picture combines the column vectors on the left side of the equations to produce the vector b on the right side.*

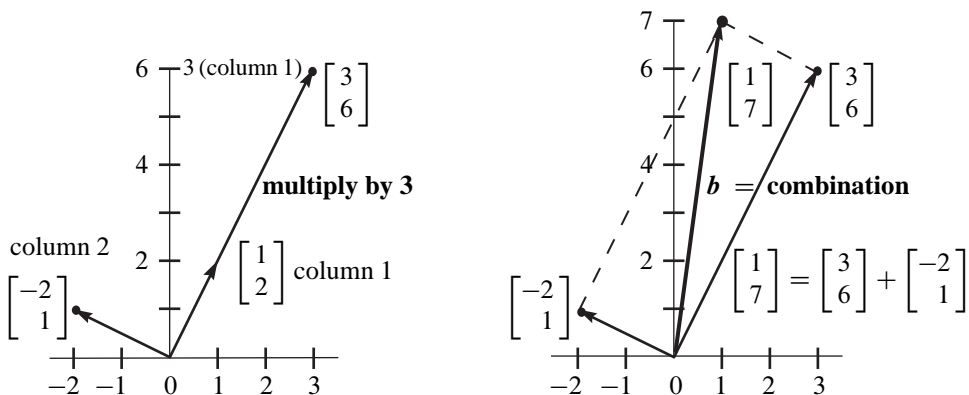


Figure 4.2: *Column picture* : A combination $3(\text{column 1}) + 1(\text{column 2})$ gives the vector b .

Figure 4.2 is the “column picture” of two equations in two unknowns. The left side shows the two separate columns, and column 1 is multiplied by 3. This multiplication by a *scalar* (a number) is one of the two basic operations in linear algebra :

Scalar multiplication
$$3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

If the components of a vector \mathbf{v} are v_1 and v_2 , then $c\mathbf{v}$ has components cv_1 and cv_2 .

The other basic operation is *vector addition*. We add the first components and the second components separately. $3 - 2$ and $6 + 1$ give the vector sum $(1, 7)$ as desired :

$$\text{Vector addition} \quad \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

The right side of Figure 4.2 shows this addition. The sum along the diagonal is the vector $\mathbf{b} = (1, 7)$ on the right side of the linear equations.

To repeat : The left side of the vector equation is a *linear combination* of the columns. The problem is to find the right coefficients $x = 3$ and $y = 1$. We are combining scalar multiplication and vector addition into one step. That combination step is crucially important, because it contains both of the basic operations on vectors : *multiply and add*.

$$\text{Linear combination of the 2 columns} \quad 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

Of course the solution $x = 3, y = 1$ is the same as in the row picture. I don't know which picture you prefer ! Two intersecting lines are more familiar at first. You may like the row picture better, but only for a day. My own preference is to combine column vectors. It is a lot easier to see a combination of four vectors in four-dimensional space, than to visualize how four "planes" might possibly meet at a point. (*Even one three-dimensional plane in four-dimensional space is hard enough. . .*)

The *coefficient matrix* on the left side of equation (1) is the 2 by 2 matrix A :

$$\text{Coefficient matrix} \quad A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

This is very typical of linear algebra, to look at a matrix by rows and also by columns. Its rows give the row picture and its columns give the column picture. Same numbers, different pictures, same equations. We write those equations as a matrix problem $A\mathbf{v} = \mathbf{b}$:

$$\text{Matrix multiplies vector} \quad \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

The row picture deals with the two rows of A . The column picture combines the columns. The numbers $x = 3$ and $y = 1$ go into the solution vector \mathbf{v} . Here is matrix-vector multiplication, matrix A times vector \mathbf{v} . Please look at this multiplication $A\mathbf{v}$!

$$\text{Dot products with rows} \quad A\mathbf{v} = \mathbf{b} \quad \text{is} \quad \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}. \quad (3)$$

Combination of columns

Linear Combinations of Vectors

Before I go to three dimensions, let me show you the most important operation on vectors. We can see a vector like $v = (3, 1)$ as a pair of numbers, or as a point in the plane, or as an arrow that starts from $(0, 0)$. The arrow ends at the point $(3, 1)$ in Figure 4.3.

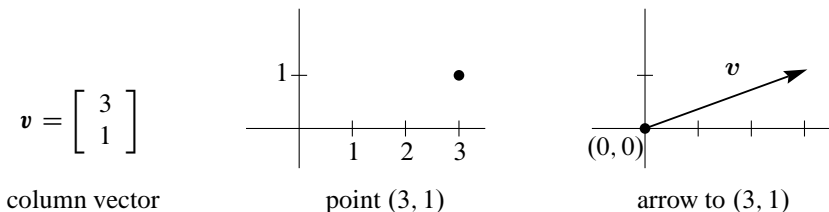


Figure 4.3: The vector v is given by two numbers or a point or an arrow from $(0, 0)$.

A first step is to multiply that vector by any number c . If $c = 2$ then the vector is doubled to $2v$. If $c = -1$ then it changes direction to $-v$. Always the “scalar” c multiplies each separate component (here 3 and 1) of the vector v . The arrow doubles the length to show $2v$ and it reverses direction to show $-v$:

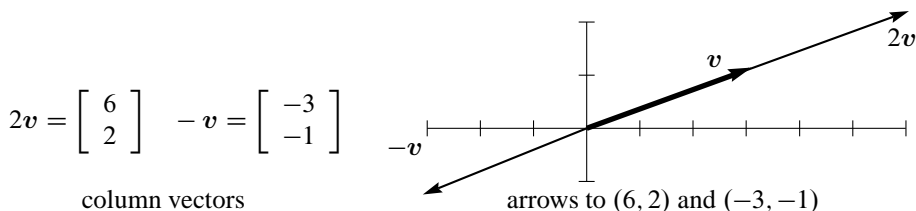


Figure 4.4: Multiply the vector $v = (3, 1)$ by scalars $c = 2$ and -1 to get $cv = (3c, c)$.

If we have another vector $w = (-1, 1)$, we can add it to v . Vector addition $v + w$ can use numbers (the normal way) or it can use the arrows (to visualize $v + w$). The arrows in Figure 4.5 go head to tail: **At the end of v , place the start of w .**

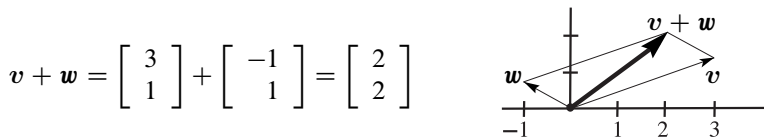


Figure 4.5: The sum of $v = (3, 1)$ and $w = (-1, 1)$ is $v + w = (2, 2)$. This is also $w + v$.

Allow me to say, adding $v + w$ and multiplying cv will soon be second nature. In themselves they are not impressive. What really counts is when you do both at once.

Multiply cv and also $d\mathbf{w}$, then add to get the **linear combination** $cv + d\mathbf{w}$.

Linear combination $2v + 3w$

$$2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

This is the basic operation of linear algebra! If you have two 5-dimensional vectors like $v = (1, 1, 1, 1, 2)$ and $w = (3, 0, 0, 1, 0)$, you can multiply v by 2 and w by 1. You can combine to get $2v + w = (5, 2, 2, 3, 4)$. Every combination $cv + d\mathbf{w}$ is a vector in the big 5-dimensional space \mathbf{R}^5 .

I admit that there is no picture to show these vectors in \mathbf{R}^5 . Somehow I imagine arrows going to v and w . If you think of all the vectors cv , they form a line in \mathbf{R}^5 . The line goes in both directions from $(0, 0, 0, 0, 0)$ because c can be positive or negative or zero.

Similarly there is a line of all vectors $d\mathbf{w}$. The hard but all-important part is to imagine all the combinations $cv + d\mathbf{w}$. Add all vectors on one line to all vectors on the other line, and what do you get? It is a “2-dimensional plane” inside the big 5-dimensional space. I don’t lose sleep trying to visualize that plane. (There is no problem in working with the five numbers.) For linear combinations in high dimensions, algebra wins.

Dot Product of v and w

The other important operation on vectors is a kind of multiplication. This is not ordinary multiplication and we don’t write $v\mathbf{w}$. The output from v and w will be one number and it is called the **dot product** $v \cdot w$.

DEFINITION The **dot product** of $v = (v_1, v_2)$ and $w = (w_1, w_2)$ is the number $v \cdot w$:

$$v \cdot w = v_1w_1 + v_2w_2. \quad (4)$$

The dot product of $v = (3, 1)$ and $w = (-1, 1)$ is $v \cdot w = (3)(-1) + (1)(1) = -2$.

Example 1 The column vectors $(1, 2)$ and $(-2, 1)$ have a *zero* dot product :

Dot product is zero

Perpendicular vectors

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2 + 2 = 0.$$

In mathematics, zero is always a special number. For dot products, it means that *these two vectors are perpendicular*. The angle between them is 90° .

The clearest example of two perpendicular vectors is $i = (1, 0)$ along the x axis and $j = (0, 1)$ up the y axis. Again the dot product is $i \cdot j = 0 + 0 = 0$. Those vectors i and j form a right angle. They are the columns of the 2 by 2 **identity matrix** I .

The dot product of $v = (3, 1)$ and $w = (1, 2)$ is 5. Soon $v \cdot w$ will reveal the angle between v and w (not 90°). Please check that $w \cdot v$ is also 5.

Multiplying a Matrix A and a Vector v

Linear equations have the form $Av = b$. The right side b is a column vector. On the left side, the coefficient matrix A multiplies the unknown column vector v (we don't use a "dot" for Av). The all-important fact is that Av is computed by *dot products in the row picture*, and Av is a **combination of the columns in the column picture**.

I put those words "combination of the columns" in boldface, because this is an essential idea that is sometimes missed. One definition is usually enough in linear algebra, but Av has two definitions—the rows and the columns produce the same output vector Av .

The rules stay the same if A has n columns a_1, \dots, a_n . Then v has n components. The vector Av is still a combination of the columns, $Av = v_1a_1 + v_2a_2 + \dots + v_na_n$. **The numbers in v multiply the columns in A .** Let me start with $n = 2$.

$$\text{By rows } Av = \begin{bmatrix} (\text{row } 1) \cdot v \\ (\text{row } 2) \cdot v \end{bmatrix} \quad \text{By columns } Av = v_1(\text{column } 1) + v_2(\text{column } 2).$$

Example 2 In equation (3) I wrote "dot products with rows" and "combination of columns." Now you know what those mean. They are the two ways to look at Av :

Dot products with rows
Combination of columns

$$\begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix} = v_1 \begin{bmatrix} a \\ c \end{bmatrix} + v_2 \begin{bmatrix} b \\ d \end{bmatrix}. \quad (5)$$

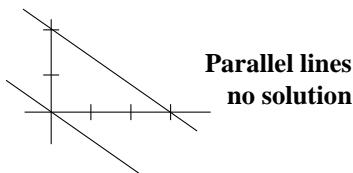
You might naturally ask, *which way to find Av ?* My own answer is this: I compute by rows and I visualize (and understand) by columns. Combinations of columns are truly fundamental. But to calculate the answer Av , I have to find one component at a time. Those components of Av are the dot products with the rows of A .

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + 3v_2 \\ 4v_1 + 5v_2 \end{bmatrix} = v_1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + v_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Singular Matrices and Parallel Lines

The row picture and column picture can fail—and they will fail together. For a 2 by 2 matrix, the row picture fails when the lines from row 1 and row 2 are parallel. The lines don't meet and $Av = b$ has no solution:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \quad \begin{aligned} 2v_1 - 3v_2 &= 6 \\ 4v_1 - 6v_2 &= 0 \end{aligned}$$



The row picture shows the problem and so does the algebra: 2 times equation 1 produces $4v_1 - 6v_2 = 12$. But equation 2 requires $4v_1 - 6v_2 = 0$. Notice that this line goes through the center point $(0, 0)$ because the right side is zero.

How does the column picture fail? *Columns 1 and 2 point in the same direction.* When the rows are “dependent”, the columns are also dependent. All combinations of the columns (2, 4) and (3, 6) lie in the same direction. Since the right side $\mathbf{b} = (6, 0)$ is not on that line, \mathbf{b} is *not* a combination of those two column vectors of A . Figure 4.6 (a) shows that there is *no solution* to the equation.

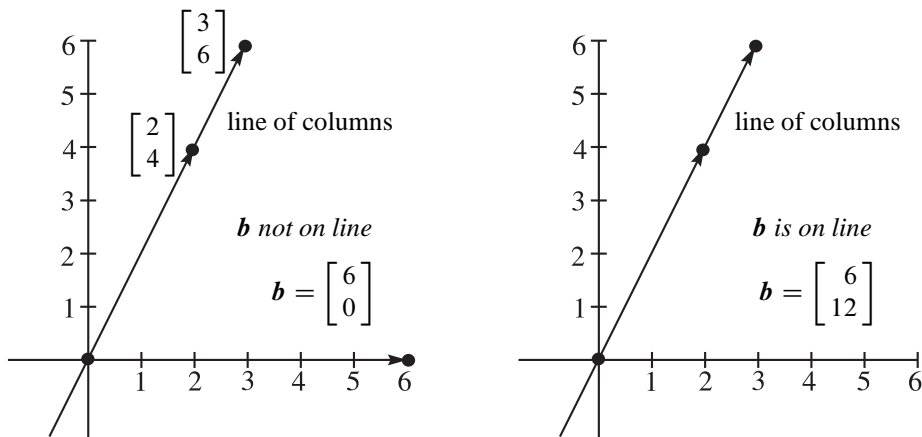
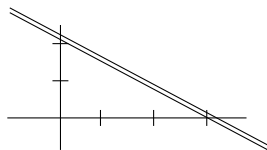


Figure 4.6: Column pictures (a) **No solution** (b) **Infinity of solutions**

Example 3 Same matrix A , now $\mathbf{b} = (6, 12)$, infinitely many solutions to $A\mathbf{v} = \mathbf{b}$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \quad \begin{array}{l} 2v_1 - 3v_2 = 6 \\ 4v_1 - 6v_2 = 12 \end{array}$$



In the row picture, the two lines are the same. *All points* on that line solve both equations. Two times equation 1 gives equation 2. Those close lines are one line.

In the column picture above, the right side $\mathbf{b} = (6, 12)$ falls right onto the line of the columns. Later we will say: \mathbf{b} is *in the column space* of A . There are infinitely many ways to produce (6, 12) as a combination of the columns. They come from infinitely many ways to produce $\mathbf{b} = (0, 0)$ (**choose any c**). Add one way to produce $\mathbf{b} = (6, 12) = 3(2, 4)$.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 3c \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 2c \begin{bmatrix} -3 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 6 \\ 12 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} -3 \\ -6 \end{bmatrix}. \quad (6)$$

The vector $\mathbf{v}_n = (3c, 2c)$ is a **null solution** and $\mathbf{v}_p = (3, 0)$ is a **particular solution**. $A\mathbf{v}_n$ equals zero and $A\mathbf{v}_p$ equals \mathbf{b} . Then $A(\mathbf{v}_p + \mathbf{v}_n) = \mathbf{b}$. Together, \mathbf{v}_p and \mathbf{v}_n give the **complete solution**, all the ways to produce $\mathbf{b} = (6, 12)$ from the columns of A :

Complete solution to $A\mathbf{v} = \mathbf{b}$ $\mathbf{v}_{\text{complete}} = \mathbf{v}_p + \mathbf{v}_n = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3c \\ 2c \end{bmatrix}. \quad (7)$

Equations and Pictures in Three Dimensions

In three dimensions, a linear equation like $x + y + 2z = 6$ produces a *plane*. The plane would go through $(0, 0, 0)$ if the right side were 0. In this case the “6” moves us to a parallel plane that misses the center point $(0, 0, 0)$.

A second linear equation will produce another plane. Normally the two planes meet in a *line*. Then a third plane (from a third equation) normally cuts through that line at a *point*. That point will lie on all three planes, so it solves all three equations.

This is the *row picture*, three planes in three-dimensional space. They meet at the solution. One big problem is that this row picture is hard to draw. Three planes are too many to see clearly how they meet (maybe Picasso could do it).

The *column picture* of $A\mathbf{v} = \mathbf{b}$ is easier. It starts with three column vectors in three-dimensional space. We want to combine those columns of A to produce the vector v_1 (column 1) + v_2 (column 2) + v_3 (column 3) = \mathbf{b} . Normally there is one way to do it. That gives the solution (v_1, v_2, v_3) — which is also the meeting point in the row picture.

I want to give an example of success (one solution) and an example of failure (no solution). Both examples are simple, but they really go deeply into linear algebra.

Example 4 Invertible matrix A , one solution \mathbf{v} for any right side \mathbf{b} .

$$A\mathbf{v} = \mathbf{b} \quad \text{is} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}. \quad (8)$$

This matrix is **lower triangular**. It has zeros above the main diagonal. Lower triangular systems are quickly solved by forward substitution, top to bottom. The top equation gives v_1 , then move down. First $v_1 = 1$. Then $-v_1 + v_2 = 3$ gives $v_2 = 4$. Then $-v_2 + v_3 = 5$ gives $v_3 = 9$.

Figure 4.7 shows the three columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. When you combine them with 1, 4, 9 you produce $\mathbf{b} = (1, 3, 5)$. In reverse, $\mathbf{v} = (1, 4, 9)$ must be the solution to $A\mathbf{v} = \mathbf{b}$.

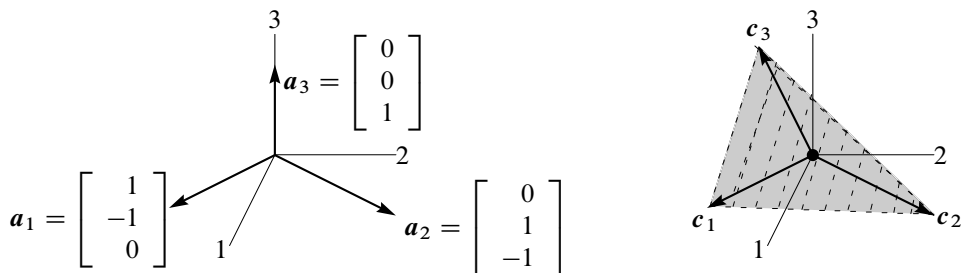


Figure 4.7: **Independent columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ not in a plane.** Dependent columns $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ are three vectors all in the same plane.

Example 5 Singular matrix: no solution to $C\mathbf{v} = \mathbf{b}$ or infinitely many solutions (depending on \mathbf{b}).

$$\begin{aligned} w_1 - w_3 &= b_1 \\ -w_1 + w_2 &= b_2 \\ -w_2 + w_3 &= b_3 \end{aligned} \quad \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}. \quad (9)$$

This matrix C is a “circulant.” The diagonals are constants, all 1’s or all 0’s or all -1 ’s. The diagonals circle around so each diagonal has three equal entries. Circulant matrices will be perfect for the Fast Fourier Transform (FFT) in Chapter 8.

To see if $C\mathbf{w} = \mathbf{b}$ has a solution, add those three equations to get $0 = b_1 + b_2 + b_3$.

$$\text{Left side} \quad (w_1 - w_3) + (-w_3 + w_2) + (-w_2 + w_3) = 0. \quad (10)$$

$C\mathbf{w} = \mathbf{b}$ cannot have a solution unless $0 = b_1 + b_2 + b_3$. The components of $\mathbf{b} = (1, 3, 5)$ do not add to zero, so $C\mathbf{w} = (1, 3, 5)$ has no solution.

Figure 4.7 shows the problem. **The three columns of C lie in a plane. All combinations $C\mathbf{w}$ of those columns will lie in that same plane.** If the right side vector \mathbf{b} is not in the plane, then $C\mathbf{w} = \mathbf{b}$ cannot be solved. The vector $\mathbf{b} = (1, 3, 5)$ is off the plane, because the equation of the plane requires $b_1 + b_2 + b_3 = 0$.

Of course $C\mathbf{w} = (0, 0, 0)$ always has the zero solution $\mathbf{w} = (0, 0, 0)$. But when the columns of C are in a plane (as here), there are additional nonzero solutions to $C\mathbf{w} = \mathbf{0}$. Those three equations are $w_1 = w_3$ and $w_1 = w_2$ and $w_2 = w_3$. The **null solutions** are $\mathbf{w}_n = (c, c, c)$. When all three components are equal, we have $C\mathbf{w}_n = \mathbf{0}$.

The vector $\mathbf{b} = (1, 2, -3)$ is also in the plane of the columns, because it does have $b_1 + b_2 + b_3 = 0$. In this good case there must be a **particular solution** to $C\mathbf{w}_p = \mathbf{b}$. There are many particular solutions \mathbf{w}_p , since any solution can be a particular solution. I will choose the particular $\mathbf{w}_p = (1, 3, 0)$ that ends in $w_3 = 0$:

$$C\mathbf{w}_p = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

The **complete solution** is
 $\mathbf{w}_{\text{complete}} = \mathbf{w}_p + \text{any } \mathbf{w}_n$

Summary These two matrices A and C , with third columns \mathbf{a}_3 and \mathbf{c}_3 , allow me to mention two key words of linear algebra: *independence and dependence*. This book will develop those ideas much further. I am happy if you see them early in the two examples:

a_1, a_2, a_3 are independent

A is invertible

$A\mathbf{v} = \mathbf{b}$ has one solution \mathbf{v}

c_1, c_2, c_3 are dependent

C is singular

$C\mathbf{w} = \mathbf{0}$ has many solutions \mathbf{w}_n

Eventually we will have n column vectors in n -dimensional space. The matrix will be n by n . The key question is whether $A\mathbf{v} = \mathbf{0}$ has only the zero solution. Then the columns don’t lie in any “hyperplane.” When columns are independent, the matrix is invertible.

Problem Set 4.1

Problems 1–8 are about the row and column pictures of $Av = b$.

- 1 With $A = I$ (the identity matrix) draw the planes in the row picture. Three sides of a box meet at the solution $v = (x, y, z) = (2, 3, 4)$:

$$\begin{array}{l} 1x + 0y + 0z = 2 \\ 0x + 1y + 0z = 3 \\ 0x + 0y + 1z = 4 \end{array} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Draw the four vectors in the column picture. Two times column 1 plus three times column 2 plus four times column 3 equals the right side b .

- 2 If the equations in Problem 1 are multiplied by 2, 3, 4 they become $DV = B$:

$$\begin{array}{l} 2x + 0y + 0z = 4 \\ 0x + 3y + 0z = 9 \\ 0x + 0y + 4z = 16 \end{array} \quad \text{or} \quad DV = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix} = B$$

Why is the row picture the same? Is the solution V the same as v ? What is changed in the column picture—the columns or the right combination to give B ?

- 3 If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the vectors in the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be $x = 2$, $x + y = 5$, $z = 4$.
- 4 Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$. Find a third point halfway between.
- 5 The first of these equations plus the second equals the third:

$$\begin{array}{l} x + y + z = 2 \\ x + 2y + z = 3 \\ 2x + 3y + 2z = 5. \end{array}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also _____. The equations have infinitely many solutions (the whole line L). Find three solutions on L .

- 6 Move the third plane in Problem 5 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—*why not?* The first two planes meet along the line L , but the third plane doesn't _____ that line.
- 7 In Problem 5 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a “singular case” because the third column is _____. Find two combinations of the columns that give $b = (2, 3, 5)$. This is only possible for $b = (4, 6, c)$ if $c =$ _____.

- 8 Normally 4 “planes” in 4-dimensional space meet at a _____. Normally 4 vectors in 4-dimensional space can combine to produce \mathbf{b} . What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $\mathbf{b} = (3, 3, 3, 2)$?

Problems 9–14 are about multiplying matrices and vectors.

- 9 Compute each $A\mathbf{x}$ by dot products of the rows with the column vector:

$$(a) \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

- 10 Compute each $A\mathbf{x}$ in Problem 9 as a combination of the columns:

$$9(a) \text{ becomes } A\mathbf{x} = 2 \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}.$$

How many separate multiplications for $A\mathbf{x}$, when the matrix is “3 by 3”?

- 11 Find the two components of $A\mathbf{x}$ by rows or by columns:

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

- 12 Multiply A times \mathbf{x} to find three components of $A\mathbf{x}$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- 13 (a) A matrix with m rows and n columns multiplies a vector with _____ components to produce a vector with _____ components.
 (b) The planes from the m equations $A\mathbf{x} = \mathbf{b}$ are in _____-dimensional space. The combination of the columns of A is in _____-dimensional space.
- 14 Write $2x + 3y + z + 5t = 8$ as a matrix A (how many rows?) multiplying the column vector $\mathbf{x} = (x, y, z, t)$ to produce \mathbf{b} . The solutions \mathbf{x} fill a plane or “hyperplane” in 4-dimensional space. *The plane is 3-dimensional with no 4D volume.*

Problems 15–22 ask for matrices that act in special ways on vectors.

- 15 (a) What is the 2 by 2 identity matrix? I times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} x \\ y \end{bmatrix}$.
 (b) What is the 2 by 2 exchange matrix? P times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} y \\ x \end{bmatrix}$.

- 16 (a) What 2 by 2 matrix R rotates every vector by 90° ? R times $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} -y \\ x \end{bmatrix}$.
 (b) What 2 by 2 matrix R^2 rotates every vector by 180° ?
- 17 Find the matrix P that multiplies (x, y, z) to give (y, z, x) . Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z) .
- 18 What 2 by 2 matrix E subtracts the first component from the second component? What 3 by 3 matrix does the same?

$$E \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad E \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}.$$

- 19 What 3 by 3 matrix E multiplies (x, y, z) to give $(x, y, z + x)$? What matrix E^{-1} multiplies (x, y, z) to give $(x, y, z - x)$? If you multiply $(3, 4, 5)$ by E and then multiply by E^{-1} , the two results are (____) and (____).
- 20 What 2 by 2 matrix P_1 projects the vector (x, y) onto the x axis to produce $(x, 0)$? What matrix P_2 projects onto the y axis to produce $(0, y)$? If you multiply $(5, 7)$ by P_1 and then multiply by P_2 , you get (____) and (____).
- 21 What 2 by 2 matrix R rotates every vector through 45° ? The vector $(1, 0)$ goes to $(\sqrt{2}/2, \sqrt{2}/2)$. The vector $(0, 1)$ goes to $(-\sqrt{2}/2, \sqrt{2}/2)$. Those determine the matrix. Draw these particular vectors in the xy plane and find R .
- 22 Write the dot product of $(1, 4, 5)$ and (x, y, z) as a matrix multiplication Av . The matrix A has one row. The solutions to $Av = \mathbf{0}$ lie on a ____ perpendicular to the vector _____. The columns of A are only in ____-dimensional space.
- 23 In MATLAB notation, write the commands that define this matrix A and the column vectors v and b . What command would test whether or not $Av = b$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad v = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

- 24 If you multiply the 4 by 4 all-ones matrix $A = \text{ones}(4)$ and the column $v = \text{ones}(4, 1)$, what is $A*v$? (Computer not needed.) If you multiply $B = \text{eye}(4) + \text{ones}(4)$ times $w = \text{zeros}(4, 1) + 2*\text{ones}(4, 1)$, what is $B*w$?

Questions 25–27 review the row and column pictures in 2, 3, and 4 dimensions.

- 25 Draw the row and column pictures for the equations $x - 2y = 0$, $x + y = 6$.
- 26 For two linear equations in three unknowns x, y, z , the row picture will show (2 or 3) (lines or planes) in (2 or 3)-dimensional space. The column picture is in (2 or 3)-dimensional space. The solutions normally lie on a _____.
- 27 For four linear equations in two unknowns x and y , the row picture shows four _____. The column picture is in ____-dimensional space. The equations have no solution unless the vector on the right side is a combination of _____.

Challenge Problems

- 28** Invent a 3 by 3 **magic matrix** M_3 with entries $1, 2, \dots, 9$. All rows and columns and diagonals add to 15. The first row could be 8, 3, 4. What is M_3 times $(1, 1, 1)$? What is M_4 times $(1, 1, 1, 1)$ if a 4 by 4 magic matrix has entries $1, \dots, 16$?
- 29** Suppose \mathbf{u} and \mathbf{v} are the first two columns of a 3 by 3 matrix A . Which third column \mathbf{w} would make this matrix singular? Describe a typical column picture of $A\mathbf{v} = \mathbf{b}$ in that singular case, and a typical row picture (for a random \mathbf{b}).
- 30** **Multiplying by A is a “linear transformation”.** Those important words mean: If \mathbf{w} is a combination of \mathbf{u} and \mathbf{v} , then $A\mathbf{w}$ is the same combination of $A\mathbf{u}$ and $A\mathbf{v}$. It is this “**linearity**” $A\mathbf{w} = cA\mathbf{u} + dA\mathbf{v}$ that gives us the name *linear algebra*. If $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A\mathbf{u}$ and $A\mathbf{v}$ are the columns of A . Combine $\mathbf{w} = c\mathbf{u} + d\mathbf{v}$. If $\mathbf{w} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ how is $A\mathbf{w}$ connected to $A\mathbf{u}$ and $A\mathbf{v}$?
- 31** A 9 by 9 **Sudoku matrix** S has the numbers $1, \dots, 9$ in every row and column, and in every 3 by 3 block. For the all-ones vector $\mathbf{v} = (1, \dots, 1)$, what is $S\mathbf{v}$? A better question is: **Which row exchanges will produce another Sudoku matrix?** Also, which exchanges of block rows give another Sudoku matrix? Section 4.5 will look at all possible permutations (reorderings) of the rows. I see 6 orders for the first 3 rows, all giving Sudoku matrices. Also 6 permutations of the next 3 rows, and of the last 3 rows. And 6 block permutations of the block rows?
- 32** Suppose the second row of A is some number c times the first row:

$$A = \begin{bmatrix} a & b \\ ca & cb \end{bmatrix}.$$

Then if $a \neq 0$, the second column of A is what number d times the first column? **A square matrix with dependent rows will also have dependent columns.** This is a crucial fact coming soon.