Chapter 2

Second Order Equations

2.1 Second Derivatives in Science and Engineering

Second order equations involve the second derivative d^2y/dt^2. Often this is shortened to y'', and then the first derivative is y'. In physical problems, y' can represent velocity v and the second derivative $y'' = a$ is **acceleration**: the rate dy'/dt that velocity is changing.

The most important equation in dynamics is Newton's Second Law $\boldsymbol{F = ma}$. Compare a second order equation to a first order equation, and allow them to be nonlinear:

$$\textbf{First order} \quad y' = f(t, y) \qquad \textbf{Second order} \quad y'' = F(t, y, y') \tag{1}$$

The second order equation needs **two initial conditions**, normally $y(0)$ and $y'(0)$—the initial velocity as well as the initial position. Then the equation tells us $y''(0)$ and the movement begins.

When you press the gas pedal, that produces acceleration. The brake pedal also brings acceleration but it is *negative* (the velocity decreases). The steering wheel produces acceleration too ! Steering changes the direction of velocity, not the speed.

Right now we stay with straight line motion and one–dimensional problems:

$$\boldsymbol{\frac{d^2y}{dt^2} > 0} \quad \text{(speeding up)} \qquad\qquad \boldsymbol{\frac{d^2y}{dt^2} < 0} \quad \text{(slowing down).}$$

The graph of $y(t)$ bends upwards for $y'' > 0$ (the right word is *convex*). Then the velocity y' (slope of the graph) is increasing. The graph bends downwards for $y'' < 0$ (*concave*). Figure 2.1 shows the graph of $\boldsymbol{y = \sin t}$, when the acceleration is $a = d^2y/dt^2 = -\sin t$. The important equation $\boldsymbol{y'' = -y}$ leads to $\sin t$ and $\cos t$.

Notice how the velocity dy/dt (slope of the graph) changes sign in between zeros of y.

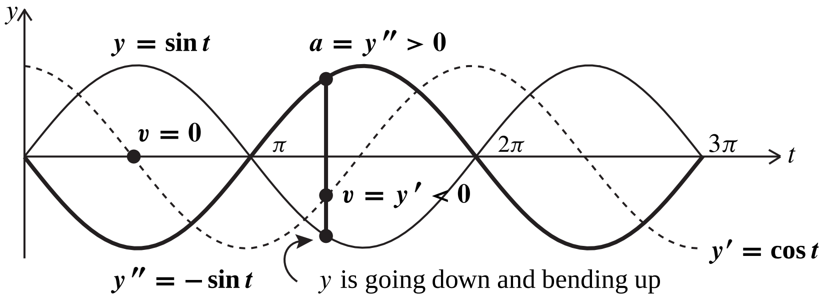

Figure 2.1: $y'' > 0$ means that velocity y' (or slope) increases. The curve bends upward.

The best examples of $F = ma$ come when the force F is $-ky$, a constant k times the "position" or "displacement" $y(t)$. This produces the oscillation equation.

Fundamental equation of mechanics $$m\frac{d^2y}{dt^2} + ky = 0 \qquad (2)$$

Think of a mass hanging at the bottom of a spring (Figure 2.2). The top of the spring is fixed, and the spring will stretch. Now stretch it a little more (move the mass downward by $y(0)$) and let go. The spring pulls back on the mass. Hooke's Law says that the force is $F = -ky$, proportional to the stretching distance y. Hooke's constant is k.

The mass will oscillate up and down. The oscillation goes on forever, because equation (2) does not include any friction (damping term $b\,dy/dt$). The oscillation is a perfect cosine, with $y = \cos\omega t$ and $\omega = \sqrt{k/m}$, because the second derivative has to produce k/m to match $y'' = -(k/m)y$.

Oscillation at frequency $\boldsymbol{\omega = \sqrt{\frac{k}{m}}}$ $$y = y(0)\cos\left(\sqrt{\frac{k}{m}}\,t\right). \qquad (3)$$

At time $t = 0$, this shows the extra stretching $y(0)$. The derivative of $\cos\omega t$ has a factor $\omega = \sqrt{k/m}$. The second derivative y'' has the required $\omega^2 = k/m$, so $my'' = -ky$.

The movement of one spring and one mass is especially simple. There is only one frequency ω. When we connect N masses by a line of springs there will be N frequencies—then Chapter 6 has to study the eigenvalues of N by N matrices.

$$m\frac{d^2y}{dt^2} = -ky$$

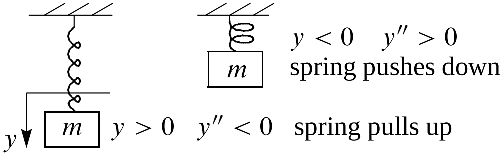

Figure 2.2: Larger k = stiffer spring = *faster* ω. Larger m = heavier mass = *slower* ω.

Initial Velocity $y'(0)$

Second order equations have *two* initial conditions. The motion starts in an initial position $y(0)$, and its initial velocity is $y'(0)$. We need both $y(0)$ and $y'(0)$ to determine the two constants c_1 and c_2 in the complete solution to $my'' + ky = 0$:

$$\textbf{“Simple harmonic motion”} \qquad y = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right). \qquad (4)$$

Up to now the motion has started from rest ($y'(0) = 0$, no initial velocity). Then c_1 is $y(0)$ and c_2 is zero: only cosines. As soon as we allow an initial velocity, the sine solution $y = c_2 \sin \omega t$ must be included. But its coefficient c_2 is not just $y'(0)$.

$$\text{At} \quad t = 0, \quad \frac{dy}{dt} = c_2\,\omega \cos \omega t \quad \text{matches} \quad y'(0) \quad \text{when} \quad \boldsymbol{c_2 = \frac{y'(0)}{\omega}}. \qquad (5)$$

The original solution $y = y(0) \cos \omega t$ matched $y(0)$, with zero velocity at $t = 0$. The new solution $y = (y'(0)/\omega) \sin \omega t$ has the right initial velocity and it starts from zero. When we combine those two solutions, $y(t)$ matches both conditions $y(0)$ and $y'(0)$:

$$\textbf{Unforced oscillation} \qquad \boldsymbol{y(t) = y(0) \cos \omega t + \frac{y'(0)}{\omega} \sin \omega t \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}}}. \qquad (6)$$

With a trigonometric identity, I can combine those two terms (cosine and sine) into one.

Cosine with Phase Shift

We want to rewrite the solution (6) as $\boldsymbol{y(t) = R\cos(\omega t - \alpha)}$. The amplitude of $y(t)$ will be the positive number R. The phase shift or lag in this solution will be the angle α. By using the right identity for the cosine of $\omega t - \alpha$, we match both $\cos \omega t$ and $\sin \omega t$:

$$R \cos(\omega t - \alpha) = R \cos \omega t \, \cos \alpha + R \sin \omega t \, \sin \alpha. \qquad (7)$$

This combination of $\cos \omega t$ and $\sin \omega t$ agrees with the solution (6) if

$$R \cos \alpha = y(0) \quad \text{and} \quad R \sin \alpha = \frac{y'(0)}{\omega}. \qquad (8)$$

Squaring those equations and adding will produce R^2:

$$\textbf{Amplitude } \boldsymbol{R} \qquad R^2 = R^2(\cos^2 \alpha + \sin^2 \alpha) = (y(0))^2 + \left(\frac{y'(0)}{\omega}\right)^2. \qquad (9)$$

The ratio of the equations (8) will produce the tangent of α:

$$\textbf{Phase lag } \boldsymbol{\alpha} \qquad \tan \alpha = \frac{R \sin \alpha}{R \cos \alpha} = \frac{y'(0)}{\omega\, y(0)}. \qquad (10)$$

Problem 14 will discuss the angle α we should choose, since different angles can have the same tangent. The tangent is the same if α is increased by π or any multiple of π.

The pure cosine solution that started from $y'(0) = 0$ has *no phase shift*: $\alpha = 0$. Then the new form $y(t) = R \cos(\omega t - \alpha)$ is the same as the old form $y(0) \cos \omega t$.

Frequency ω or f

If the time t is measured in *seconds*, the frequency ω is in *radians per second*. Then ωt is in radians—it is an angle and $\cos \omega t$ is its cosine. But not everyone thinks naturally about radians. Complete cycles are easier to visualize. So frequency is also measured in *cycles per second*. A typical frequency in your home is $f = 60$ cycles per second. One cycle per second is usually shortened to $\boldsymbol{f = 1}$ **Hertz**. A complete cycle is 2π radians, so *$f = 60$ Hertz is the same frequency as $\omega = 120\pi$ radians per second.*

The **period** is the time T for one complete cycle. Thus $T = 1/f$. This is the only page where f is a frequency—on all other pages $f(t)$ is the driving function.

Frequency $\boldsymbol{\omega = 2\pi f}$ **Period** $\boldsymbol{T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}}$.

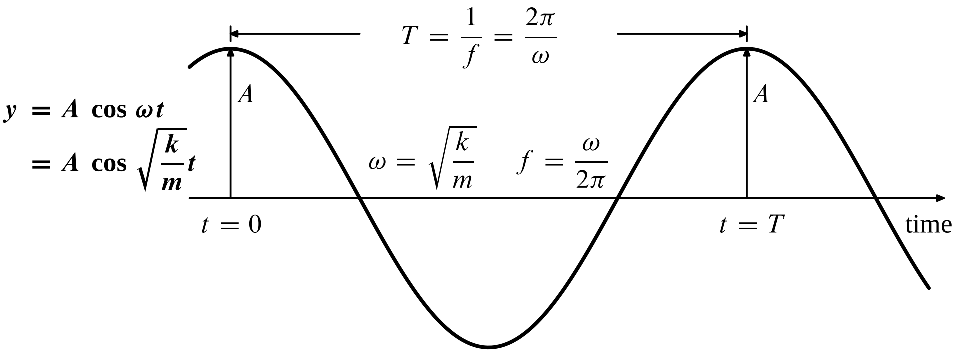

Figure 2.3: Simple harmonic motion $y = A \cos \omega t$: amplitude A and frequency ω.

Harmonic Motion and Circular Motion

Harmonic motion is up and down (or side to side). **When a point is in circular motion, its projections on the x and y axes are in harmonic motion**. Those motions are closely related, which is why a piston going up and down can produce circular motion of a flywheel. The harmonic motion "speeds up in the middle and slows down at the ends" while the point moves with constant speed around the circle.

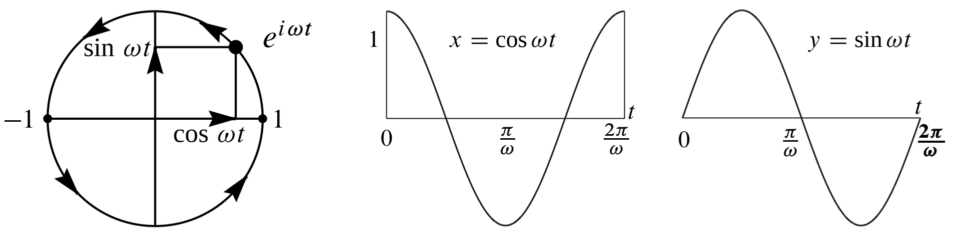

Figure 2.4: Steady motion around a circle produces cosine and sine motion along the axes.

Response Functions

I want to introduce some important words. The **response** is the output $y(t)$. Up to now the only inputs were the initial values $y(0)$ and $y'(0)$. In this case $y(t)$ would be the *initial value response* (but I have never seen those words). When we only see a few cycles of the motion, initial values make a big difference. In the long run, what counts is the response to a *forcing function* like $f = \cos\omega t$.

Now ω is the **driving frequency** on the right hand side, where the **natural frequency** $\omega_n = \sqrt{k/m}$ is decided by the left hand side: ω comes from y_p, ω_n comes from y_n. When the motion is driven by $\cos\omega t$, a particular solution is $y_p = Y\cos\omega t$:

Forced motion $y_p(t)$ at frequency ω
$$my'' + ky = \cos\omega t \qquad \boldsymbol{y_p(t) = \frac{1}{k - m\omega^2}\cos\omega t}. \tag{11}$$

To find $y_p(t)$, I put $Y\cos\omega t$ into $my'' + ky$ and the result was $(k - m\omega^2)Y\cos\omega t$. This matches the driving function $\cos\omega t$ when $Y = 1/(k - m\omega^2)$.

The initial conditions are nowhere in equation (11). Those conditions contribute the null solution y_n, which oscillates at the natural frequency $\omega_n = \sqrt{k/m}$. Then $k = m\omega_n^2$.

If I replace k by $m\omega_n^2$ in the response $y_p(t)$, I see $\omega_n^2 - \omega^2$ in the denominator:

Response to $\cos\omega t$
$$y_p(t) = \frac{1}{m\left(\omega_n^2 - \omega^2\right)}\cos\omega t. \tag{12}$$

Our equation $my'' + ky = \cos\omega t$ has no damping term. That will come in Section 2.3. It will produce a phase shift α. Damping will also reduce the amplitude $|Y(\omega)|$. The amplitude is all we are seeing here in $Y(\omega)\cos\omega t$:

Frequency response
$$Y(\omega) = \frac{1}{k - m\omega^2} = \frac{1}{m\left(\omega_n^2 - \omega^2\right)}. \tag{13}$$

The mass and spring, or the inductance and capacitance, decide the natural frequency ω_n. The response to a driving term $\cos\omega t$ (or $e^{i\omega t}$) is multiplication by the frequency response $Y(\omega)$. *The formula changes when $\omega = \omega_n$—we will study resonance*!

With damping in Section 2.3, the frequency response $Y(\omega)$ will be a complex number. We can't escape complex arithmetic and we don't want to. The magnitude $|Y(\omega)|$ will give the **magnitude response** (or amplitude response). The angle θ in the complex plane will decide the **phase response** (then $\alpha = -\theta$ because we measure the phase lag).

The response is $Y(\omega)e^{i\omega t}$ to $f(t) = e^{i\omega t}$ and the response is $g(t)$ to $f(t) = \delta(t)$. These show the frequency response Y from equation (13) and the impulse response g from equation (15). *$Ye^{i\omega t}$ and $g(t)$ are the two key solutions to $my'' + ky = f(t)$.*

Impulse Response = Fundamental Solution

The most important solution to a linear differential equation will be called $g(t)$. In mathematics g is the *fundamental solution*. In engineering g is the *impulse response*. It is a particular solution when the right side $f(t) = \delta(t)$ is an impulse (a delta function).

The same $g(t)$ solves $mg'' + kg = 0$ when the initial velocity is $g'(0) = 1/m$.

Fundamental solution	$\boldsymbol{mg'' + kg = \delta(t)}$	**with zero initial conditions**	(14)
Null solution also	$\boldsymbol{g(t) = \dfrac{\sin \omega_n t}{m\omega_n}}$	**has $\boldsymbol{g(0)=0}$ and $\boldsymbol{g'(0)=\dfrac{1}{m}}$.**	(15)

To find that null solution, I just put its initial values 0 and $1/m$ into equation (6). The cosine term disappeared because $g(0) = 0$.

I will show that those two problems give the same answer. Then this whole chapter will show why $g(t)$ is so important. For first order equations $y' = ay + q$ in Chapter 1, the fundamental solution (impulse response, growth factor) was $g(t) = e^{at}$. The first two names were not used, but you saw how e^{at} dominated that whole chapter.

I will first explain the response $g(t)$ in physical language. *We strike the mass and it starts to move*. All our force is acting at one instant of time: *an impulse*. A finite force within one moment is impossible for an ordinary function, only possible for a delta function. Remember that the integral of $\delta(t)$ jumps to 1 when we pass the point $t = 0$.

If we integrate $mg'' = \delta(t)$, nothing happens before $t = 0$. In that instant, the integral jumps to 1. The integral of the left side mg'' is mg'. Then $mg' = 1$ instantly at $t = 0$. This gives $g'(0) = 1/m$. You see that computing with an impulse $\delta(t)$ needs some faith.

The point of $\boldsymbol{g(t)}$ is that it solves the equation for any forcing function $\boldsymbol{f(t)}$:

$$\boldsymbol{my'' + ky = f(t)} \text{ has the particular solution } \boldsymbol{y(t) = \int_0^t g(t-s)\,f(s)\,ds}. \qquad (16)$$

That was the key formula of Chapter 1, when $g(t - s)$ was $e^{a(t-s)}$ and the equation was first order. Section 2.3 will find $g(t)$ when the differential equation includes damping. The coefficients in the equation will stay constant, to allow a neat formula for $g(t)$.

You may feel uncertain about working with delta functions—a means to an end. We will verify this final solution $y(t)$ in three different ways:

1 Substitute $y(t)$ from (16) directly into the differential equation (Problem 21)

2 Solve for $y(t)$ by variation of parameters (Section 2.6)

3 Solve again by using the Laplace transform $Y(s)$ (Section 2.7).

■ REVIEW OF THE KEY IDEAS ■

1. $my'' + ky = 0$: A mass on a spring oscillates at the natural frequency $\omega_n = \sqrt{k/m}$.
2. $my'' + ky = \cos\omega t$: This driving force produces $y_p = (\cos\omega t)/m\left(\omega_n^2 - \omega^2\right)$.
3. There is resonance when $\omega_n = \omega$. The solution $y_p = t\sin\omega t$ includes a new factor t.
4. $mg'' + kg = \delta(t)$ gives $\boldsymbol{g(t) = (\sin\omega_n t)/m\omega_n}$ = null solution with $g'(0) = 1/m$.
5. Fundamental solution g : Every driving function f gives $y(t) = \int_0^t g(t-s)\,f(s)\,ds$.
6. Frequency : ω radians per second or f cycles per second (f Hertz). Period $T = 1/f$.

Problem Set 2.1

1 Find a cosine and a sine that solve $d^2y/dt^2 = -9y$. This is a second order equation so we expect *two constants* C and D (from integrating twice) :

Simple harmonic motion $\quad y(t) = C\cos\omega t + D\sin\omega t.\quad$ What is ω ?

If the system starts from rest (this means $dy/dt = 0$ at $t = 0$), which constant C or D will be zero ?

2 In Problem 1, which C and D will give the starting values $y(0) = 0$ and $y'(0) = 1$?

3 Draw Figure 2.3 to show simple harmonic motion $y = A\cos(\omega t - \alpha)$ with phases $\alpha = \pi/3$ and $\alpha = -\pi/2$.

4 Suppose the circle in Figure 2.4 has radius 3 and circular frequency $f = 60$ Hertz. If the moving point starts at the angle $-45°$, find its x-coordinate $A\cos(\omega t - \alpha)$. The phase lag is $\alpha = 45°$. When does the point first hit the x axis ?

5 If you drive at 60 miles per hour on a circular track with radius $R = 3$ miles, what is the time T for one complete circuit ? Your circular frequency is $f =$ ______ and your angular frequency is $\omega =$ ______ (with what units ?). The period is T.

6 The total energy E in the oscillating spring-mass system is

$$E = \textbf{kinetic}\text{ energy in mass} + \textbf{potential}\text{ energy in spring} = \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{k}{2}y^2.$$

Compute E when $y = C\cos\omega t + D\sin\omega t$. The energy is constant !

7 Another way to show that the total energy E is constant :

Multiply $\boldsymbol{my'' + ky = 0}$ by $\boldsymbol{y'}$. Then integrate $my'y''$ and kyy'.

8 A **forced oscillation** has another term in the equation and in the solution:

$$\frac{d^2y}{dt^2} + 4y = F\cos\omega t \quad \text{has} \quad y = C\cos 2t + D\sin 2t + A\cos\omega t.$$

(a) Substitute y into the equation to see how C and D disappear (they give y_n). Find the forced amplitude A in the particular solution $y_p = A\cos\omega t$.

(b) In case $\omega = 2$ (forcing frequency = natural frequency), what answer does your formula give for A? The solution formula for y breaks down in this case.

9 Following Problem 8, write down the complete solution $y_n + y_p$ to the equation

$$m\frac{d^2y}{dt^2} + ky = F\cos\omega t \text{ with } \omega \neq \omega_n = \sqrt{k/m} \text{ (no resonance).}$$

The answer y has free constants C and D to match $y(0)$ and $y'(0)$ (A *is fixed* by F).

10 Suppose Newton's Law $F = ma$ has the force F in the *same* direction as a:

$$my'' = +ky \qquad \text{including} \qquad y'' = 4y.$$

Find two possible choices of s in the exponential solutions $y = e^{st}$. The solution is not sinusoidal and s is real and the oscillations are gone. Now y is unstable.

11 Here is a *fourth order* equation: $d^4y/dt^4 = 16y$. Find *four* values of s that give exponential solutions $y = e^{st}$. You could expect four initial conditions on y: $y(0)$ is given along with what three other conditions?

12 To find a particular solution to $y'' + 9y = e^{ct}$, I would look for a multiple $y_p(t) = Ye^{ct}$ of the forcing function. What is that number Y? When does your formula give $Y = \infty$? (Resonance needs a new formula for Y.)

13 In a particular solution $y = Ae^{i\omega t}$ to $y'' + 9y = e^{i\omega t}$, what is the amplitude A? The formula blows up when the forcing frequency ω = what natural frequency?

14 Equation (10) says that the tangent of the phase angle is $\tan\alpha = y'(0)/\omega y(0)$. First, check that $\tan\alpha$ is dimensionless when y is in meters and time is in seconds. Next, if that ratio is $\tan\alpha = 1$, should you choose $\alpha = \pi/4$ or $\alpha = 5\pi/4$? Answer:

$$\textbf{Separately you want } R\cos\alpha = y(0) \textbf{ and } R\sin\alpha = y'(0)/\omega.$$

If those right hand sides are positive, choose the angle α between 0 and $\pi/2$.
If those right hand sides are negative, add π and choose $\alpha = 5\pi/4$.

Question: If $y(0) > 0$ and $y'(0) < 0$, does α fall between $\pi/2$ and π or between $3\pi/2$ and 2π? If you plot the vector from $(0, 0)$ to $(y(0), y'(0)/\omega)$, its angle is α.

15 Find a point on the sine curve in Figure 2.1 where $y > 0$ but $v = y' < 0$ and also $a = y'' < 0$. The curve is sloping down and bending down.

Find a point where $y < 0$ but $y' > 0$ and $y'' > 0$. The point is below the x-axis but the curve is sloping ______ and bending ______.

16 (a) Solve $y'' + 100y = 0$ starting from $y(0) = 1$ and $y'(0) = 10$. (**This is y_n**.)

(b) Solve $y'' + 100y = \cos\omega t$ with $y(0) = 0$ and $y'(0) = 0$. (**This can be y_p**.)

17 Find a particular solution $y_p = R\cos(\omega t - \alpha)$ to $y'' + 100y = \cos\omega t - \sin\omega t$.

18 Simple harmonic motion also comes from a linear pendulum (like a grandfather clock). At time t, the height is $A\cos\omega t$. What is the frequency ω if the pendulum comes back to the start after 1 second? The period does not depend on the amplitude (a large clock or a small metronome or the movement in a watch can all have $T = 1$).

19 If the phase lag is α, what is the time lag in graphing $\cos(\omega t - \alpha)$?

20 What is the response $y(t)$ to a delayed impulse if $my'' + ky = \delta(t - T)$?

21 (Good challenge) Show that $y = \int_0^t g(t-s)\,f(s)\,ds$ has $my'' + ky = f(t)$.

1 Why is $y' = \int_0^t g'(t-s)\,f(s)\,ds + g(0)\,f(t)$? Notice the two t's in y.

2 Using $g(0) = 0$, explain why $y'' = \int_0^t g''(t-s)\,f(s)\,ds + g'(0)\,f(t)$.

3 Now use $g'(0) = 1/m$ and $mg'' + kg = 0$ to confirm $my'' + ky = f(t)$.

22 With $f = 1$ (direct current has $\omega = 0$) verify that $my'' + ky = 1$ for this y:

$$\textbf{Step response} \quad y(t) = \int_0^t \frac{\sin\omega_n(t-s)}{m\omega_n}\,1\,ds = y_p + y_n \ \text{ equals } \ \frac{\mathbf{1}}{\boldsymbol{k}} - \frac{\mathbf{1}}{\boldsymbol{k}}\cos\boldsymbol{\omega_n t}.$$

23 (Recommended) For the equation $d^2y/dt^2 = 0$ find the null solution. Then for $d^2g/dt^2 = \delta(t)$ find the fundamental solution (start the null solution with $g(0) = 0$ and $g'(0) = 1$). For $y'' = f(t)$ find the particular solution using formula (16).

24 For the equation $d^2y/dt^2 = e^{i\omega t}$ find a particular solution $y = Y(\omega)e^{i\omega t}$. Then $Y(\omega)$ is the frequency response. Note the "resonance" when $\omega = 0$ with the null solution $y_n = 1$.

25 Find a particular solution $Ye^{i\omega t}$ to $my'' - ky = e^{i\omega t}$. The equation has $-ky$ instead of ky. What is the frequency response $Y(\omega)$? For which ω is Y infinite?