1. (10 pts. T Oct 20) Do the following problems from Apostol §4.9

(a) #4. $f'(x) = 2x + a$, so at $x = 2$, the slope of $f$ is $4 + a$. Since the slope of $y = 2x$ is 2, we want $a = -2$. For $f$ to pass through $(2, 4)$, $b$ must be 4.

#10. If $c = 0$ then

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > 0 \\ a & \text{if } x = 0. \end{cases}$$

This function is never continuous at 0 for any choice of $a$, so it is also never differentiable at 0. If $c < 0$, $f(x)$ is not defined at $x = 0$.

Thus assume $c > 0$. In order for $f'(c)$ to exist $f(x)$ must be continuous at $c$. Thus

$$a + bc^2 = \frac{1}{c}. \tag{1}$$

The equality

$$\lim_{x \to c^+} f'(x) = \lim_{x \to c^-} f'(x) \quad \text{gives us}$$

$$-\frac{1}{c^2} = 2bc, \quad \text{so} \quad b = -\frac{1}{2c^3}. \tag{2}$$

Plugging this into (1) gives

$$a - \frac{1}{2c} = \frac{1}{c}, \quad \text{so} \quad a = \frac{3}{2c}.$$

(b) #15 and #16.

#15a. True. Let $\Delta x = h - a$. Then $h \to a \Rightarrow \Delta x \to 0$, and $h = a + \Delta x$, so

$$\lim_{h \to a} \frac{f(h) - f(a)}{h - a} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = f'(a).$$

#15b. True.

$$\lim_{h \to 0} \frac{f(a) - f(a - h)}{h} = \lim_{h \to 0} \frac{f(a - h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = f'(a).$$
#15c.
\[
\lim_{t \to 0} \frac{f(a + 2t) - f(a)}{t} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h/2} = 2f'(a) \quad \text{by setting } 2t = h.
\]

If \(f'(a) = 0\), then \(2f'(a) = f'(a)\) so the statement is true. If \(f'(a) \neq 0\), then \(2f'(a) \neq f'(a)\), so the statement is false. Therefore if we consider, for instance, \(f(x) = x\), the statement fails.

#15d.
\[
\lim_{t \to 0} \frac{f(a + 2t) - f(a + t)}{2t} = \lim_{h \to 0} \frac{f(a + h) - f(a + \frac{h}{2})}{h} \quad \text{by setting } h=2t
\]
\[
= \lim_{h \to 0} \frac{f(a + h) - f(a + \frac{h}{2})}{h} + \frac{f(a) - f(a)}{h} = f'(a) - \lim_{h \to 0} \frac{f(a + \frac{h}{2}) - f(a)}{h}
\]
\[
= f'(a) - \frac{1}{2}f'(a) = \frac{1}{2}f'(a).
\]

(c) #16

#16a.
\[
D^*((f \cdot g)(x)) = \lim_{h \to 0} \frac{[f \cdot g](x + h) - [f \cdot g](x)}{h} = \lim_{h \to 0} \frac{f^2(x + h)g^2(x + h) - f^2(x)g^2(x)}{h}
\]

Adding and subtracting \(\lim_{h \to 0} \frac{f^2(x + h)g^2(x)}{h}\) and regrouping gives
\[
= \lim_{h \to 0} \frac{f^2(x + h)[g^2(x + h) - g^2(x)]}{h} + g^2(x) \left[ \lim_{h \to 0} \frac{f^2(x + h) - f^2(x)}{h} \right]
\]
\[
= f^2(x) \cdot \left[ \lim_{h \to 0} \frac{g^2(x + h) - g^2(x)}{h} \right] + g^2(x) \left[ \lim_{h \to 0} \frac{f^2(x + h) - f^2(x)}{h} \right]
\]
\[
= f^2 D^*g + g^2 D^*f
\]

#16b.
\[
D^*(f(x)) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{[f(x + h) - f(x)][f(x + h) + f(x)]}{h}
\]
\[
= \left[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \right] \cdot \left[ \lim_{h \to 0} f(x + h) + f(x) \right] = f'(x)[2f(x)].
\]

#16c. \(D^* = Df \Rightarrow 2f'(x)f(x) = f'(x) \Rightarrow f'(x) = 0 \) or \(f(x) = \frac{1}{2}x\). Thus any constant function or function of the term \(f(x) = \frac{1}{2}x + b\) will have this property.
2. (12 pts. T Oct 20) Prove that if \( f'(x) = 0 \), then \( f(x) = c \), for some \( c \in \mathbb{R} \).

See proof of Theorem 4.7 in Apostol.

3. (12 pts. R Oct 22) Do the following problems from Apostol §4.6

(a) (1 pt. each) #2, #9, #23. - See Apostol.

(b) (1 pt.) \( \frac{d}{dx} \sin(2009) = 0 \).

(c) (4 pts. each) #24, #38. Let \( f(x) \) be the solution to 38a. In 38b, write your solution in terms of \( f(x) \) and \( f'(x) \).

#24. Write \( g_n = f_1 \cdots f_n \). We will show that

\[
g'_n = \sum_{i=1}^{n} f_1 \cdots f_{i-1} \cdot f'_i \cdot f_{i+1} \cdots f_n.
\]

When \( n = 1 \), we have \( g_1 = f_1 \), so \( g'_1 = f'_1 \). Now assume the equation holds for \( n \) and consider \( g_{n+1} \). We know \( g_{n+1} = g_n \cdot f_{n+1} \), so by the product rule,

\[
g'_{n+1} = g'_n f_{n+1} + g_n f'_{n+1}
\]

\[
= \sum_{i=1}^{n} (f_1 \cdots f_{i-1} \cdot f'_i \cdot f_{i+1} \cdots f_n) f_{n+1} + (f_1 \cdots f_n) f'_{n+1}
\]

\[
= \sum_{i=1}^{n+1} f_1 \cdots f_{i-1} \cdot f'_i \cdot f_{i+1} \cdots f_n.
\]

#38a. \( 0 + 1 + 2x + \ldots + nx^{n-1} = \frac{d}{dx} (1 + x + \ldots + x^n) = \frac{d}{dx} \left( \frac{x^{n+1} - 1}{x - 1} \right) = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} \).

#38b. Let \( f(x) = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} \). Then

\[
1^2 x + 2^2 x^2 + \ldots + n^2 x^n = x \cdot \frac{d}{dx} \left[ x \left( 1 + 2x + 3x^2 + \ldots + nx^n \right) \right]
= x \left( 1 + 2x + \ldots + nx^n \right) + xf'(x)
= xf(x) + x^2 f'(x).
\]

4. (6 pts. R Oct 22) Do the following problems from Apostol §4.12

(a) (2 pts.) #16. - See Apostol.

(b) (2 pts. each) #19a, #19d. - See Apostol
**Bonus.** Suppose that $f$ is continuous on $[a, b]$, and $f^{(n)}$ is differentiable on $[a, b]$ for all $n \leq p + q$ (i.e. the $p + q + 1$st derivative of $f$ exists). Assume that

$$
\begin{align*}
    f(a) &= f'(a) = \ldots = f^{(p)}(a) = 0 \\
    f(b) &= f'(b) = \ldots = f^{(q)}(b) = 0.
\end{align*}
$$

Prove there is some $c \in (a, b)$ such that $f^{(p+q+1)}(c) = 0$, for $p = 1$ and $q = 2$.

Since we know that

$$
f(a) = f(b) = 0,
$$

we also know, by Rolle’s Theorem, that there is some $c \in (a, b)$ such that $f'(c) = 0$. Now we have

$$
f'(a) = f'(c) = f'(b) = 0
$$

By Rolle’s Theorem again, we have $d_1 \in (a, c)$ and $d_2 \in (c, b)$ such that

$$
f''(d_1) = f''(d_2) = f''(b) = 0.
$$

Thus by Rolle’s Theorem, there exist $e_1 \in (d_1, d_2)$ and $e_2 \in (d_2, b)$ such that

$$
f^{(3)}(e_1) = f^{(3)}(e_2) = 0.
$$

One final application of Rolle’s Theorem gives us some $r \in (e_1, e_2)$ such that

$$
f^{(4)}(r) = 0.
$$

Since $4 = 1 + 2 + 1$, we are done.