1. (R Oct 15)
   (a) Verify that multiplication of rational numbers is well-defined.

   Let \( \frac{p}{q} \sim \frac{p'}{q'} \) and \( \frac{m}{n} \sim \frac{m'}{n'} \). Then we want to show that
   \[ \frac{p}{q} \cdot \frac{m}{n} \sim \frac{p'}{q'} \cdot \frac{m'}{n'}, \]
   i.e., \( \frac{pm}{qn} \sim \frac{p'm'}{q'n'} \).

   We know that \( pq' = p'q \) and \( mn' = nm' \). Multiplying these two equations gives
   \( (pq')(mn') = (p'm')(qn) \).

   Thus \( \frac{pm}{qn} \sim \frac{p'm'}{q'n'} \).

   (b) Prove that if \( r, s \in \mathbb{Q} \) and if \( a \in \mathbb{R}^+ \), then \( (a^r)^s = a^{rs} \).

   See Notes G.

2. (R Oct 15) Let \( a, b \in \mathbb{Z} \) and \( n \in \mathbb{Z}^+ \).
   (a) Show that the relation below is an equivalence relation:

   \( a \sim b \iff \text{there exists } q \in \mathbb{Z} \text{ such that } a = b + nq \)

   i. \( a = a + n \cdot 0 \), so \( \sim \) is reflexive.
   ii. \( a = b + nq \Rightarrow b = a + n(-q) \), so \( \sim \) is symmetric.
   iii. \( a = b + nq \) and \( b = c + nq' \Rightarrow a = c + n(q + q') \), so \( \sim \) is transitive.

   (b) Define \( \overline{a} + \overline{b} = \frac{a}{b} \). Show that this addition is well-defined.

   We want to show that if \( a \sim a' \) and \( b \sim b' \) then \( a + b \sim a' + b' \). But \( a \sim a' \) and \( b \sim b' \)
   mean that there exist \( q, p \in \mathbb{Z} \) such that \( a = a' + nq \) and \( b = b' + np \). Therefore
   \[ a + b = (a' + nq) + (b' + np) = (a' + b') + n(q + p), \]
   so \( a + b \sim a' + b' \). Notice, by the way, that we would get the same additive structure
   if we defined \( \overline{a} + \overline{b} = \{x + y| x \in \overline{a}, y \in \overline{b}\} \).
(c) Define $\bar{a} \cdot \bar{b} = \overline{a \cdot b}$. Show this is well-defined too.

Assume $\exists q$ and $p$ such that $a = a' + nq$ and $b = b' + np$. Then $a \cdot b \sim a' \cdot b'$, since

\[
a \cdot b = (a' + nq)(b' + np) = a'b' + a'nq + b'np + n^2pq = (a'b') + n(a'p + b'q + npq).
\]

(d) Consider the case $n = 2$ and write out an addition and multiplication table. Do the same for the case $n = 3$. Show that $\mathbb{Z}/4\mathbb{Z}$ is not a field by writing out a multiplication table and finding an element without a multiplicative inverse.

For typesetting purposes, we will write 0, 1, 2 and 3 in place of 0, 1, 2 and 3.

\[
\begin{array}{c|cc}
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 \\
2 & 0 & 2 & 0 & 2 \\
3 & 0 & 3 & 2 & 1 \\
\end{array}
\]

For the case $n = 4$, the element $\bar{2}$ does not have a multiplicative inverse!

3. (F Oct 16)

(a) The function $f(x) = \frac{1}{x}$ is continuous on $(0, 1]$, and $\sin x$ is continuous everywhere. We saw in class that if $u(x)$ is continuous at $p$ and $v(x)$ is continuous at $u(p)$, then $v(u(x))$ is continuous at $p$. Thus $\sin(\frac{1}{x})$ is continuous on $(0, 1]$.

(b) We want to show that for any $\epsilon, \delta \in \mathbb{R}^+$, there exists $p \in (0, 1]$ such that $f(V) \not\subseteq U$ (as usual, define $V = (p - \delta, p + \delta)$ and $U = (f(p) - \epsilon, f(p) + \epsilon)$). As we saw in lecture, if $f(p - \delta) - f(p + \delta) > 2\epsilon$, then $f(V) \not\subseteq U$.

Given any $\epsilon$ and $\delta \in \mathbb{R}^+$, let $p < \sqrt{\frac{\delta}{2\epsilon} + \frac{\delta^2}{4}}$. Then

\[
f\left(p - \frac{\delta}{2}\right) - f\left(p + \frac{\delta}{2}\right) = \frac{1}{p - \delta/2} - \frac{1}{p + \delta/2} = \frac{\delta}{p^2 - \delta^2/4} > \frac{\delta}{\delta^2/2 + \frac{\delta^2}{4} - \frac{\delta^2}{4}} = 2\epsilon,
\]

so $f(p - \frac{\delta}{2})$ and $f(p + \frac{\delta}{2})$ cannot both be in $U$.

(c) We want to show that there exists an $\epsilon \in \mathbb{R}^+$ such that for any $\delta \in \mathbb{R}^+$, there exists a $p \in (0, 1]$ such that $f(V) \not\subseteq U$, for $V = (p - \delta, p + \delta)$ and $U = (f(p) - \epsilon, f(p) + \epsilon)$.

Let $\epsilon < 2$ and $\delta \in \mathbb{R}^+$. Observe that if $\tilde{V} \supseteq (a, b)$, and $b - a \geq 2\pi$, then $\sin(\tilde{V})$ must contain both $-1$ and $1$ and hence cannot be contained in $U$. 

Let $p < \sqrt{\frac{\delta}{2\pi} + \frac{\delta^2}{4}}$. Then

$$\left| f\left(p - \frac{\delta}{2}\right) - f\left(p + \frac{\delta}{2}\right) \right| = \frac{\delta}{p^2 - \frac{\delta^2}{4}} > \frac{\delta}{\frac{\delta}{2\pi} + \frac{\delta^2}{4} - \frac{\delta^2}{4}} = 2\pi.$$  

By the intermediate value theorem, $f(V) \supseteq (f(p + \frac{\delta}{2}), f(p - \frac{\delta}{2}))$, so by the observation above, $\sin(f(V))$ contains both $-1$ and $1$. Thus $\sin(f(V)) \not\subseteq U$.

4. (T Oct 20)

(a) Consider $f(x) = \frac{1}{x}$ on the interval $[-1, 1]$. Then $f$ is clearly not bounded. It is continuous on $(0, 1]$ because $g(x) = x$ is continuous and nonzero on $(0, 1]$. Similarly, $f$ is continuous on $[-1, 0)$. Therefore $f$ is piecewise continuous on $[-1, 1]$.

**Bonus.** You’ll need the derivative in this problem. Hopefully we’ll get to it by Tuesday; if we don’t, what you already know should suffice. Let $f(x) = \sqrt{x}$. Show that $f$ is uniformly continuous on $[0, \infty)$. You may assume without proof that if $d, a, b \in \mathbb{R}$, then

$$f'(x) < d \text{ on } [a, b] \Rightarrow \frac{f(b) - f(a)}{b - a} < d. \tag{1}$$

Divide the domain $[0, \infty)$ into $[0, 1]$ and $[1, \infty)$. Since $[0, 1]$ is closed and $f$ is continuous on $[0, 1]$, $f$ must also be uniformly continuous on $[0, 1]$. We will show that $f$ is also uniformly continuous on $[1, \infty)$. Once we do so, for every $\epsilon \in \mathbb{R}^+$ we will get two deltas: $\delta_{[0,1]}$ and $\delta_{[1,\infty)}$. We can set $\delta = \min\{\delta_{[0,1]}, \delta_{[1,\infty)}\}$; then for every $p \in [1, \infty)$, the map $f$ will send the neighborhood $V_p = (p - \delta, p + \delta)$ into $U_{f(p)} = (f(p) - \epsilon, f(p) + \epsilon)$.

We now show that $f$ is uniformly continuous on $[1, \infty)$. Fix $\epsilon$ and let $\delta = \epsilon$. We know that for all $x \geq 1$,

$$f'(x) = \frac{1}{2}x^{-\frac{3}{2}} < 1,$$

and that $f$ is strictly increasing. Thus if $p$ is any point in $[1, \infty)$, then

$$p \leq x < p + \delta \Rightarrow f(p) \leq f(x) < f(p + \delta).$$

But

$$\frac{f(p + \delta) - f(p)}{\delta} < 1 \Rightarrow f(p + \delta) - f(p) < \epsilon$$

$$\Rightarrow f(p) < f(p + \delta) < f(p) + \epsilon \Rightarrow f([p, p + \delta]) \subseteq U_p.$$

Similarly, $f((p - \delta, p]) \subseteq U_p$.

**Note:** The equation (1) is essentially the mean value theorem.