2. (15 pts. T Sep 22) Suppose \( f \) is integrable on \([a, b]\) and \( g \) is a function on \([a, b]\) such that \( g(x) = f(x) \) for all except a finite number of \( x \in [a, b] \). Prove that \( g \) is integrable and 
\[
\int_a^b f(x)dx = \int_a^b g(x)dx.
\]
Start by considering the case in which \( g \) differs from \( f \) at only one point, \( z \). Let \( s \) and \( t \) be step functions satisfying the Riemann Condition for some \( \varepsilon \in \mathbb{R}^+ \). Let \( P = \{x_0, \ldots, x_n\} \) be the partition for \( s \). Define a new step function \( \tilde{s} \) relative to the partition \( \tilde{P} = P \cup \{z\} \) with
\[
\tilde{s}(x) = \begin{cases} 
  s(x) & \text{if } x \neq z \\
  g(z) & \text{if } x = z.
\end{cases}
\]
Clearly \( \tilde{s}(x) \leq g(x) \) on \([a, b]\). We can define a step function \( \tilde{t}(x) \) such that \( g(x) \leq \tilde{t}(x) \) in the same way. Since changing the value of a step function at one point does not change its integral, (see §1.12),
\[
\int_a^b \tilde{s}(x)dx = \int_a^b s(x)dx, \\
\int_a^b \tilde{t}(x)dx = \int_a^b t(x)dx,
\]

hence \( \int_a^b \tilde{t}(x) - \int_a^b \tilde{s}(x) \leq \varepsilon \). Therefore \( g \) is integrable and 
\[
\int_a^b g(x)dx = \int_a^b f(x)dx.
\]
Now consider the case where \( g \) differs from \( f \) at \( n \) points, \( \{z_1, \ldots, z_n\} \), and assume that the statement holds for all functions that differ from \( f \) at \( n - 1 \) points. Define
\[
\tilde{g}(x) = \begin{cases} 
  g(x) & \text{if } x \neq z_n \\
  f(x) & \text{if } x = z_n.
\end{cases}
\]
This function agrees with \( f \) at all but \( n - 1 \) points, hence \( \int_a^b \tilde{g} = \int_a^b f \). It differs from \( g \) at only one point, hence \( \int_a^b \tilde{g} = \int_a^b g \).

4. (20 pts. F Sep 25) Compute the following:

(b) \( \int_1^3 (3x^2 + 1)dx = \left( x^3 + x \right) \bigg|_1^3 = 30 - 2 = 28 \).

(c) \( \int_{-50}^{49} (x + 50)^{100}dx = \int_0^1 x^{100}dx = \frac{x^{101}}{101} \bigg|_0^1 = \frac{1}{101} \).
(d) \[ \int_0^2 |(x-1)(3x-1)|. \] Define
\[
f(x) = \begin{cases} 
(x-1)(3x-1) & \text{if } x \in [0, \frac{1}{3}] \cup [1, 2] \\
-(x-1)(3x-1) & \text{if } x \in [\frac{1}{3}, 1]. 
\end{cases}
\]
Then \( f(x) = |(x-1)(3x-1)| \) on \([0, 2]\). Therefore,
\[
\int_0^2 |(x-1)(3x-1)| \, dx = \int_0^2 f(x) \, dx
\]
\[
= \int_0^{\frac{1}{3}} (3x^2 - 4x + 1) \, dx - \int_1^{\frac{1}{3}} (3x^2 - 4x + 1) \, dx + \int_{\frac{1}{3}}^2 (3x^2 - 4x + 1) \, dx.
\]
Let \( g(x) = x^3 - 2x^2 + x \). Then the above integral becomes
\[
g\left(\frac{1}{3}\right) - g(0) - g(1) + g\left(\frac{1}{3}\right) + g(2) - g(1) = 2g\left(\frac{1}{3}\right) - 2g(1) - g(0) + g(2).
\]

5. (10 pts.) Denote by \( r \) the radius of the sphere. Then the volume of the napkin holder is
\[
2 \int_0^h \pi (\sqrt{r^2 - x^2})^2 \, dx - 2 \int_0^h \pi (\sqrt{r^2 - h^2})^2 \, dx
\]
(the first integral is the volume of the sphere with the left and right ends cut off; the second integral is the volume of the inner cylinder). Simplifying gives
\[
2\pi r^2 h - \frac{2}{3} \pi h^3 - 2\pi r^2 h + 2\pi h^3 = \frac{4}{3} \pi h^3.
\]

**Bonus.** Find a sequence of functions \( q_n \) and a function \( q \), on \([0, 1]\), such that

(a) The \( q_n \)'s “approach” \( q \) as \( n \) gets large. More precisely, for all \( x \in [0, 1] \) there exists \( N \in \mathbb{Z}^+ \) such that \( q_n(x) = q(x) \) for all \( n > N \).

(b) \( q_n \) is integrable on \([0, 1]\).

(c) \( q \) is not integrable on \([0, 1]\).

Define \( q(x) \) as suggested by the hint.

Since the rationals are countable, so are the rationals that lie on the unit interval. We therefore have a bijection from \( \mathbb{N} \) to \( \mathbb{Q} \); denote by \( r_n \) the image of \( n \). Then \( \mathbb{Q} \cap [0, 1] = \{r_1, r_2, r_3, \ldots\} \). Define
\[
q_n(x) = \begin{cases} 
1 & \text{if } x \in \{r_1, \ldots, r_n\} \\
0 & \text{otherwise}.
\end{cases}
\]

We now prove that \( q_n \) and \( q \) satisfy the above conditions.

(a) We find for each \( x \in [0, 1]\) an \( N \) satisfying the above condition.

Case 1 If \( x \) is irrational, then \( q_n(x) = q(x) \) for all \( n \in \mathbb{N} \) (hence \( N = 0 \)).

Case 2 Suppose that \( x \) is rational. Then \( x = r_N \) for some \( N \in \mathbb{N} \), and
\[
q_n(r_N) = 1 = q(r_N) \quad \text{for all } n \geq N.
\]

(b) Each \( q_n \) differs from the function \( f(x) = 0 \) at a finite number of points and is therefore integrable.

(c) We saw on Practice Exam 1 that \( q \) is not integrable.