18.014: Fall 2009  
Homework 5  
50 pts. Due 1 pm Thursday October 15 in Room 2-108


1. (15 pts.) Let $S, T \subseteq \mathbb{R}$. ($S$ and $T$ need not be finite.) Consider the following properties:

(i) For all $s \in S$ and $t \in T$, $s < t$.

(ii) For all $s \in S$, there exists a $t \in T$ such that $s < t$.

(iii) There exists a $t \in T$ such that for all $s \in S$, $s < t$.

For each ordered pair of items (6 in all) either show that one follows from the other or construct a counter-example. For example, either prove that (i) $\Rightarrow$ (ii) or give sets $S$ and $T$ which satisfy (i) but not (ii).

[Hint: Let $X = \emptyset$.]

TRUE STATEMENT: For all $x \in X$, $x = 1$. Consider the set $S = \{x \in X | x = 1\}$. $S$ is the empty set, so $S = X$. Therefore all elements of $X$ are contained in $S$, so the statement is true. (Note: it is also true that for all $x \in X$, $x \neq 1$)

FALSE STATEMENT: There exists $x \in X$ such that $x = 1$. “There exists $x \in X$” is already false without the “such that” part.]

2. (9 pts. T Oct 6) Prove from the definition of the limit that

$$\lim_{x \to p^-} f(x) = \lim_{x \to p^+} f(x) = A \Leftrightarrow \lim_{x \to p} f(x) = A.$$

3. (11 pts. R Oct 8) Show that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. That is, there is a point $p \in [0, 1]$ with $f(p) = p$. The two dimensional version of this problem can be used to prove that at every given moment, there are two spots on earth with the exact same temperature and pressure!

4. (5 pts. R Oct 8) Show by example that the conclusion of the intermediate value theorem can fail if $f$ is only continuous on $[a, b)$ and bounded on $[a, b]$.

(a) Show that the function $\tilde{f}(x) = 2x - 5$ is unbounded.

(b) Show that $f$ is unbounded above by showing it is greater than $2x - 5$.

5. (10 pts. F Oct 9) Let $f(x) = 2x^5 - 5x^4 + 5$ for $x \geq 3$. We will show later that $f$ is strictly increasing (since its derivative is positive for $x > 2$). However, for the moment you may assume this fact without proof.

(a) Show that the function $\tilde{f}(x) = 2x - 5$ is unbounded.

(b) Show that $f$ is unbounded above by showing it is greater than $2x - 5$. 
(c) What is the domain of its inverse function $g$? [Note: A famous theorem of Modern Algebra states that it is not possible to express $g$ in terms of algebraic operations and radicals.]

**Bonus.** A function, $f : [a, b] \to \mathbb{R}$ is called Lipschitz if there exists some $k \in \mathbb{R}^+$ such that for all $x, y \in [a, b]$,

$$|f(x) - f(y)| < k|x - y|.$$

Being Lipschitz means that the function isn’t too stretchy. Is every function that is Lipschitz necessarily continuous? How about the reverse? Give a proof or counter example.