18.014: Fall 2009
Midterm Exams

Exam 1
1. (20 pts.) Suppose that \( \int_{2}^{3} e^{x^2} \, dx = A \) and \( \int_{2}^{4} e^{x^2} \, dx = B \).
   Use the properties of the integral to compute the value of \( \int_{-4}^{-3} e^{x^2} \, dx \).

2. (20 pts.) Compute \( \int_{0}^{3} 2x \lfloor \sqrt{x} \rfloor \), where \( \lfloor z \rfloor \) is the greatest integer less than or equal to \( z \).

3. (40 pts.)
   (a) State the Riemann condition for the existence of the integral \( \int_{a}^{b} f \), where \( f \) is a function on \([a, b] \).
   (b) Let \( X \subseteq \mathbb{R} \) be a set with the following property: every open interval \((c, d) \subseteq \mathbb{R} \)
      contains at least one point that is in \( X \) and one point that is not in \( X \). Let
      \[
      f(x) = \begin{cases} 
      x + 1 & x \in X \\
      x - 1 & x \notin X 
      \end{cases}
      \]
      Show that \( f(x) \) is not integrable on \([0, 1] \).

4. (40 pts.) The well-ordering principle says that every nonempty subset \( S \subseteq \mathbb{Z}^+ \) has a least element. You may use the well-ordering principle or any of the theorems on the photocopied page to complete the problems which follow.

   In the steps below you will prove that for every \( x \in \mathbb{R} \), there is exactly one integer \( \lceil x \rceil \)
   such that \( \lceil x \rceil - 1 < x \leq \lceil x \rceil \).

   Do not use the floor function or decimal expansions.
   (a) Prove that for every \( x \in \mathbb{R} \), there is exactly one smallest integer greater than or equal to \( x \). (Observe that you are proving that the ceiling function \( \lceil \cdot \rceil \) is actually a function here. A mathematician would say that you are proving that the ceiling function is “well defined”).
   (b) Prove that \( \lceil x \rceil - 1 < x \).
   (c) Prove that if \( m \) is an integer with \( m > \lceil x \rceil \), then \( m - 1 \geq x \).

   BONUS (10 pts. No partial credit for this problem. Only attempt if you have extra time!)
   Suppose that \( f \) is a bounded function defined on \([0, 1] \) such that \( f \) is integrable on every open interval \((a, b) \) for \( a, b \in (0, 1) \). Is \( f \) integrable? Give a proof or a counterexample.

Exam 2
1. (16 pts.) State the theorems below. Be sure to state the hypotheses.
   (a) Mean value theorem for derivatives.
(b) First fundamental theorem of calculus.

2. (40 pts.) True/False. For false statements, no credit without a valid counter-example. (If your counter-example involves a function, be sure to specify the domain and range.) If a statement is true, you do not have to justify your answer. True statements are not worth as many points as false statements.

(a) ____ If \( f \) is uniformly continuous on a set \( S \subseteq \mathbb{R} \), then \( f \) is continuous at every \( x \in S \).

(b) ____ If \( f \) is continuous at every \( x \in S \subseteq \mathbb{R} \), then \( f \) is uniformly continuous on \( S \).

(c) ____ Suppose that \( f \) is a function that is continuous on some set \( S \subseteq \mathbb{R} \). Then \( f \) attains its maximum and minimum on \( S \). That is, there exist \( x_m, x_M \in S \) such that \( f(x_m) \leq f(x) \leq f(x_M) \) for all \( x \in S \).

(d) ____ If a function is continuous at \( c \in \mathbb{R} \) it must also be differentiable at \( c \).

(e) ____ Every strictly increasing function from \([0, 1]\) to \([0, 1]\) has an inverse. [Recall that a function has an inverse if it is both injective and surjective.]

(f) ____ If a function is differentiable at \( c \in \mathbb{R} \) it must also be continuous at \( c \).

(g) ____ If a function \( f : [a, b] \to \mathbb{R} \) is Lipschitz, then it is uniformly continuous. [Recall that \( f \) is Lipschitz if there exists some \( k \in \mathbb{R}^+ \) such that for all \( x, y \in [a, b] \), we have the inequality \( |f(x) - f(y)| < k|x - y| \).]

(h) ____ If \( f \) is continuous on \([a, b]\) then \( \int_a^b f(x) \, dx \) exists.

(i) ____ Let \( f \) be integrable on \([a, b]\) and define \( A(x) = \int_a^x f(t) \, dt \). Then \( A(x) \) is continuous.

(j) ____ Suppose \( f \) and \( A(x) \) are as above. Then \( A(x) \) is differentiable on \([a, b]\).

3. (40 pts.) In this problem, you cannot receive full credit unless you show your work. Suppose \( f \) is a function defined and continuous for all \( x \in \mathbb{R} \), and that

\[
\begin{align*}
  f(1) &= 2 & \text{and} & \quad f(2) &= 3 & \text{and} & \quad f(3) &= 7; \\
  f'(1) &= 5 & \text{and} & \quad f'(2) &= 7 & \text{and} & \quad f'(3) &= 2; \\
  f''(1) &= 1 & \text{and} & \quad f''(2) &= 3 & \text{and} & \quad f''(3) &= 1.
\end{align*}
\]

Let \( h(x) = f(f(x)) \); compute the following values.

(a) Write your answer here: \( h'(1) = \) ________

(b) Write your answer here: \( h''(1) = \) ________

Suppose that \( f \) is invertible and let \( g(x) \) be its inverse. Compute:

(c) Write your answer here: \( g(3) = \) ________

(d) Write your answer here: \( g'(3) = \) ________

[Table from previous page:]

\[
\begin{align*}
  f(1) &= 2 & \text{and} & \quad f(2) &= 3 & \text{and} & \quad f(3) &= 7; \\
  f'(1) &= 5 & \text{and} & \quad f'(2) &= 7 & \text{and} & \quad f'(3) &= 2; \\
  f''(1) &= 1 & \text{and} & \quad f''(2) &= 3 & \text{and} & \quad f''(3) &= 1.
\end{align*}
\]
In the next part, if you use any major theorems or properties, cite them. Let \( A(x) = \int_x^{x+2} [f(t)]^2 \, dt \).

(e) Then \( A'(1) = \) __________

Exam 3

1. (14 pts.) Let \( F(x) = \int_0^x \sin(t^2) \, dt \). Without attempting to express this function in terms of the elementary functions (it can’t be done), answer the following. For each, show work or indicate reasoning.
   (a) Find the smallest positive relative maximum point \( x_m > 0 \) for \( F(x) \), indicating how you know it is a maximum point.
   (b) Express the value of \( \int_1^2 \sin(9u^2) \, du \) in terms of values of \( F(x) \).

2. (18 pts.)
   (a) Using L’Hopital’s Rule, compute \( \lim_{x \to 0^+} \frac{\ln(x+1) - x + \frac{x^2}{2}}{x^2} \).
   (b) Now we will compute the same limit using Taylor’s Formula:
      i. Compute the second Taylor polynomial for \( \ln(x+1) \).
      ii. Compute Lagrange’s form of the second error term \( E_2 \ln(x+1) \).
      iii. Now recompute the above limit using parts i and ii.

3. (22 pts.) Compute the integrals below. For (b) and (c) give your answer in terms of \( x \).
   (a) Find \( a, b, k, n, \) and \( m \) such that \( \int_0^1 x^3 \sqrt{4 - x^2} \, dx = k \int_a^b \sin^n t \cos^m t \, dt \).
   (b) \( \int \sqrt{x} \ln x \, dx = \) __________
   (c) \( \int \frac{1 + e^x}{1 - e^x} \, dx = \) __________
      [Hint: Do a substitution. You may end up with something like \( du = f(u)dx \) where the \( f(u) \) doesn’t cancel with anything. That’s ok; plug in \( dx = \frac{du}{f(u)} \) and do the resulting partial fractions integral. When you are done don’t forget to revert to \( x \).]

4. (22 pts.) Prove that the sequence \( \{0, 1, 0, 1, \ldots \} \) diverges.

5. (24 pts.) Prove that if \( f : \mathbb{R}^+ \to \mathbb{R} \) is increasing invertible function with \( \lim_{x \to \infty} f(x) = \infty \), and \( f^{-1} : \mathbb{R} \to \mathbb{R}^+ \) is the inverse of \( f \), then \( \lim_{y \to \infty} f^{-1}(y) = \infty \).

Here is the beginning of your proof:
Since \( f \) is increasing, \( f^{-1} \) is as well.
(a) WTS: \( \forall M \in \mathbb{R}^+, \exists Y \in \mathbb{R} \) such that \( \forall y \ldots \) [finish the sentence below].
(b) Know: \( \forall M' \in \mathbb{R}^+, \exists X' \in \mathbb{R} \) such that \( \forall x \ldots \) [finish the sentence below].
Now let \( M' = f(M) \); we obtain a corresponding \( X' \ldots \) [finish the rest.]
Exam 3 Make-Up

Part I

1. (21 pts.) Compute the following integrals. Give your answer in terms of $x$.

   (a) $\int \frac{1}{\sqrt{9-x^2}} \, dx$
   (b) $\int \frac{dx}{(\sqrt{1+x})^5 - (\sqrt{1+x})^3}$
   (c) $\int \frac{1}{\sqrt{1+e^x}} \, dx$

2. (24 pts.) Suppose that $\lim_{x \to a^+} g(x)$ is finite and nonzero, $\lim_{x \to a^+} h(x) = 0$, and $h(x) > 0$ for all $x \in (a, a + b)$, for some $b > 0$. Prove that

   $\lim_{x \to a^+} \frac{g(x)}{h(x)} = \infty$.

Part II

3. (29 pts.) In this problem, we will graph the function $f(x) = \frac{e^x}{x^n}$, for some $n \in \mathbb{Z}^+$. You may assume without proof that you can do a change of variables within a limit.

   (a) (16 pts.) Compute:

   (i) $\lim_{x \to \infty} \frac{e^x}{x^n}$
   (ii) $\lim_{x \to -\infty} \frac{e^x}{x^n}$
   (iii) $\lim_{x \to 0^+} \frac{e^x}{x^n}$
   (iv) $\lim_{x \to 0^-} \frac{e^x}{x^n}$

   (b) (9 pts.) Compute the critical points of $f$, the regions in which it is increasing and decreasing, and the regions in which it is concave and convex.

   (c) (4 pts.) Draw a graph of $f$. Be sure to include labels.

4. (8 pts.) Let $f(x) = (1 + x)^r$.

   (a) Find the third Taylor polynomial of $f$ for $a = 0$.

   (b) Find Lagrange’s form for the error term $E_3 f(x)$.

5. (18 pts.) Prove that if a sequence converges, then it must be bounded as well.