1. (14 pts.) Let \( F(x) = \int_0^x \sin(t^2) \, dt \). Without attempting to express this function in terms of the elementary functions (it can’t be done), answer the following. For each, show work or indicate reasoning.

(a) Find the smallest positive relative maximum point \( x_m > 0 \) for \( F(x) \), indicating how you know it is a maximum point.

\( F(x) \) has its relative extrema at points where \( F'(x) = 0 \). By the First Fundamental Theorem of Calculus, \( F'(x) = \sin(x^2) \). So

\[ F'(x) = 0 \Rightarrow \sin(x^2) = 0 \]
\[ \Rightarrow x^2 = n\pi \Rightarrow x = \pm \sqrt{n\pi}, \text{ for } n \in \mathbb{Z}. \]

A relative extremum \( \alpha \) is a maximum if \( F''(\alpha) < 0 \) (since \( F'' \) is continuous). Since \( F''(x) = 2x \cos(x^2) \) is negative when \( x = \sqrt{\pi} \), the first positive relative maximum point is \( x_m = \sqrt{\pi} \).

(b) Express the value of \( \int_1^2 \sin(9u^2) \, du \) in terms of values of \( F(x) \).

We substitute \( t = 3u \). Then \( dt = 3 \, du \); If \( u = 1 \), then \( t = 3 \), and if \( u = 2 \), then \( t = 6 \). Therefore

\[
\int_1^2 \sin(9u^2) \, du = \int_3^6 \frac{1}{3} \sin(t^2) \, dt \\
= \frac{1}{3} \int_0^6 \sin(t^2) \, dt - \frac{1}{3} \int_0^3 \sin(t^2) \, dt \\
= \frac{1}{3} F(6) - \frac{1}{3} F(3).
\]

2. (18 pts.)

(a) Using L’Hopital’s Rule, compute \( \lim_{x \to 0^+} \frac{\ln(x + 1) - x + \frac{x^2}{2}}{x^2} \)

Denote by \( L \) the limit in question. Note that the numerator and denominator approach 0, so we can use L’Hopital’s Rule. The above limit becomes

\[
L = \lim_{x \to 0^+} \frac{\frac{1}{x+1} - 1 + x}{2x}
\]
Again, the numerator and denominator approach 0, so we can use L'Hopital's Rule again. Thus

\[ L = \lim_{x \to 0^+} \frac{-1}{(x+1)^2} + \frac{1}{2} = \frac{-1 + 1}{2} = 0 \]

(b) Now we will compute the same limit using Taylor's Formula:

i. Compute the second Taylor polynomial for \( \ln(x + 1) \).

Let \( f(x) = \ln(x + 1) \). Then \( f'(x) = \frac{1}{x+1} \), and \( f''(x) = \frac{-1}{(x+1)^2} \). The second Taylor polynomial is

\[ T_2 f(x) = f(0) + f'(0) \frac{x}{1!} + f''(0) \frac{x^2}{2!} \]

\[ = \ln(1) + \left( \frac{1}{1} \right) x + \left( \frac{-1}{1} \right) \frac{x^2}{2} \]

\[ = x - \frac{x^2}{2}. \]

ii. Compute Lagrange's form of the second error term \( E_2 \ln(x + 1) \).

First we compute the third derivative of \( f \): \( f^{(3)}(x) = \frac{2}{(x+1)^3} \).

\[ E_2 f(x) = f^{(3)}(c_x) \frac{x^3}{3!} = \frac{2}{(c_x+1)^3} \frac{x^3}{6} = \frac{x^3}{3(c_x+1)^3}, \quad \text{where} \ c_x \in [0, x] \]

iii. Now recompute the above limit using parts i and ii.

\[ \lim_{x \to 0^+} \frac{\ln(x + 1) - x + \frac{x^2}{2}}{x^2} = \lim_{x \to 0^+} \frac{x - \frac{x^2}{2} + \frac{x^3}{3(c_x+1)^3} - x + \frac{x^2}{2}}{x^2} \]

\[ = \lim_{x \to 0^+} \frac{x}{x^2} \frac{1}{3(c_x+1)^3} \]

\[ = 0 \]

3. (22 pts.) Compute the integrals below. For (b) and (c) give your answer in terms of \( x \).

(a) Find \( a, b, k, n, \) and \( m \) such that \( \int_0^1 x^3 \sqrt{4 - x^2} \, dx = k \int_a^b \sin^n t \cos^m t \, dt \).

We substitute \( x = 2 \sin(t) \). Then \( dx = 2 \cos(t) \, dt \); if \( x = 0 \), then \( t = 0 \) and if \( x = 1 \), then \( \sin(t) = \frac{1}{2} \), so \( t = \frac{\pi}{6} \). The above integral now becomes

\[ \int_0^1 x^3 \sqrt{4 - x^2} \, dx = \int_0^\frac{\pi}{6} (8 \sin^3 t) \sqrt{4 - 4 \sin^2 t} (2 \cos t \, dt) \]

\[ = 32 \int_0^\frac{\pi}{6} \sin^3 t \cos^2 t \, dt. \]
Therefore, \( a = 0, b = \frac{\pi}{6}, k = 32, n = 3, \) and \( m = 2. \)
(Note that \( \sqrt{4 - 4 \sin^2 t} = 2 \left| \cos t \right| \). However, on the interval \([0, \frac{\pi}{6}]\), \( \cos t \) is positive, so we can drop the absolute values.)

(b) We integrate by parts, since we can see that taking the derivative of \( \ln x \) will yield \( \frac{1}{x} \), which will cancel with some of the other \( x \)'s.

\[
\int \sqrt{x} \ln x \, dx = \frac{2x^{3/2}}{3} \ln x - \frac{2}{3} \int x^{3/2} \frac{1}{x} \, dx
= \frac{2^{3/2}}{3} \ln x - \frac{2}{3} \int x^{1/2} \, dx
= \frac{2x^{3/2}}{3} \ln x - \frac{4}{9} x^{3/2} + C.
\]

(c) Let \( I = \int \frac{1 + e^x}{1 - e^x} \, dx \). Let \( u = e^x \). Then \( du = e^x \, dx = u \, dx \). Plugging this in gives

\[
\int \frac{1 + u}{(1 - u)u} \, du = \int \left[ \frac{A}{1 - u} + \frac{B}{u} \right] \, du
= \int \left[ \frac{2}{1 - u} + \frac{1}{u} \right] \, du
= -2 \ln |1 - u| + \ln |u| + C
= -2 \ln |1 - e^x| + \ln |e^x| + C
\]

4. (22 pts.) Prove that the sequence \( \{0, 1, 0, 1, \ldots \} \) diverges.
Let

\[
a_n = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
1 & \text{if } n \text{ is even}
\end{cases}
\]

Suppose \( a_n \) converged to \( a \in \mathbb{R} \).
Choose \( \epsilon = \frac{1}{4} \). Then, by the definition of convergence, there is a corresponding \( N \in \mathbb{Z}^+ \), such that for all \( n > N \), we have \( |a_n - a| < \frac{1}{4} \). Thus

\[
|a_{N+1} - a| < \frac{1}{4} \quad \text{and} \quad |a_{N+2} - a| < \frac{1}{4}.
\]

Using the triangle inequality, we have

\[
|a_{N+1} - a_{N+2}| = |(a_{N+1} - a) - (a - a_{N+2})|
\leq |a_{N+1} - a| + |a_{N+2} - a| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]

But since \( a_{N+1} \) and \( a_{N+2} \) are consecutive, one of them is 1 and the other is 0, so

\[
|a_{N+1} - a_{N+2}| = 1.
\]
This contradicts (1).
Thus \( \{a_n\} \) does not converge, i.e. the sequence \( \{0, 1, 0, 1, \ldots\} \) diverges.

5. **(24 pts.)** Prove that if \( f : \mathbb{R}^+ \to \mathbb{R} \) is increasing invertible function with \( \lim_{x \to \infty} f(x) = \infty \), and \( f^{-1} : \mathbb{R} \to \mathbb{R}^+ \) is the inverse of \( f \), then \( \lim_{y \to \infty} f^{-1}(y) = \infty \).

(a) WTS: \( \forall M \in \mathbb{R}^+, \exists Y \in \mathbb{R} \) such that \( \forall y > Y, f^{-1}(y) > M \).
(b) Know: \( \forall M' \in \mathbb{R}^+, \exists X' \in \mathbb{R} \) such that \( \forall x > X', f(x) > M' \).

Fix \( M \). Now let \( M' = f(M) \); we obtain a corresponding \( X' \). Now let \( Y = f(X') \). Take any \( y > Y \). Since \( f \) is increasing, \( f^{-1} \) is as well, so
\[
y > Y \quad \Rightarrow \quad f^{-1}(y) > f^{-1}(Y) = X'.
\]

But we know that for every \( x > X' \), we have \( f(x) > M' \). So \( f(f^{-1}(y)) > M' \). Therefore \( y > M' \). Now, since \( f^{-1} \) is increasing, this tells us that
\[
f^{-1}(y) > f^{-1}(M') = M.
\]