18.014: Fall 2009
Exam 3

You may cite without proof anything from lecture, the book, the homework, or this exam (provided it does not trivialize the problem). A correct answer without proper justification will not necessarily receive credit.

1. (14 pts.) Let \( F(x) = \int_0^x \sin(t^2) \, dt \). Without attempting to express this function in terms of the elementary functions (it can’t be done), answer the following. For each, show work or indicate reasoning.

   (a) Find the smallest positive relative maximum point \( x_m > 0 \) for \( F(x) \), indicating how you know it is a maximum point.

   (b) Express the value of \( \int_1^2 \sin(9u^2) \, du \) in terms of values of \( F(x) \).
2. (18 pts.)

(a) Using L’Hopital’s Rule, compute \( \lim_{x \to 0^+} \frac{\ln(x + 1) - x + \frac{x^2}{2}}{x^2} \)

(b) Now we will compute the same limit using Taylor’s Formula:
   i. Compute the second Taylor polynomial for \( \ln(x + 1) \).

   ii. Compute Lagrange’s form of the second error term \( E_2 \ln(x + 1) \).

   iii. Now recompute the above limit using parts i and ii.
3. (22 pts.) Compute the integrals below. For (b) and (c) give your answer in terms of x.

(a) Find $a, b, k, n, m$ such that $\int_0^1 x^3 \sqrt{4 - x^2} \, dx = k \int_a^b \sin^n t \cos^m t \, dt$.

(b) $\int \sqrt{x} \ln x \, dx = \frac{x^{3/2} \ln x}{3} - \frac{x^{3/2}}{9} + C$
(c) \[ \int \frac{1 + e^x}{1 - e^x} \, dx = \] 

[Hint: Do a substitution. You may end up with something like \( du = f(u) \, dx \) where the \( f(u) \) doesn’t cancel with anything. That’s ok; plug in \( dx = \frac{du}{f(u)} \) and do the resulting partial fractions integral. When you are done don’t forget to revert to \( x \).]

4. (22 pts.) Prove that the sequence \( \{0, 1, 0, 1, \ldots\} \) diverges.
5. **(24 pts.)** Prove that if $f : \mathbb{R}^+ \to \mathbb{R}$ is increasing invertible function with $\lim_{x \to \infty} f(x) = \infty$, and $f^{-1} : \mathbb{R} \to \mathbb{R}^+$ is the inverse of $f$, then $\lim_{y \to \infty} f^{-1}(y) = \infty$.

Here is the beginning of your proof:
Since $f$ is increasing, $f^{-1}$ is as well.
(a) WTS: $\forall M \in \mathbb{R}^+, \exists Y \in \mathbb{R}$ such that $\forall y \ldots$ [finish the sentence below].

(b) Know: $\forall M' \in \mathbb{R}^+, \exists X' \in \mathbb{R}$ such that $\forall x \ldots$ [finish the sentence below].

Now let $M' = f(M)$; we obtain a corresponding $X'\ldots$ [finish the rest.]