1. **(15 pts)** Prove from the axioms for a field that \( a \cdot (-b) = (-a) \cdot b = -(a \cdot b) \). 

2. (a) **(3 pts)** What is the definition of \( f'(x) \), for some \( x \in \mathbb{R} \)?

   (b) **(12 pts)** The function \( f(p) \) below is an example of a function that is differentiable everywhere but whose derivative is not continuous everywhere.

   \[
   f(x) = \begin{cases} 
   x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\
   0 & \text{if } x = 0 
   \end{cases}
   \]

   i. Compute \( f'(0) \).
   
   ii. Compute \( f'(x) \) for \( x \neq 0 \).
   
   iii. Explain why \( f'(x) \) is not continuous at 0.

3. (a) **(3 pts)** State the extreme value theorem.

   (b) **(3 pts)** State the intermediate value theorem.

Here is a lemma that might be useful in solving part (c) below:

**Lemma 1** If \( f \) is integrable on \([a, b]\) and

\[
m \leq f(x) \leq M, \quad \text{for all } x \in [a, b],
\]

then there exists a real number \( n \in [m, M] \) such that

\[
\int_a^b f(x)dx = (b - a) \cdot n.
\]

**Proof:** By the comparison theorem (Apostol Theorem 1.5),

\[
\int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx.
\]

Thus

\[
(b - a)m \leq \int_a^b f(x)dx \leq (b - a)M.
\]

Dividing through by \((b - a)\) gives

\[
m \leq \frac{\int_a^b f(x)dx}{b - a} \leq M.
\]

Now let

\[
n = \frac{\int_a^b f(x)dx}{b - a}.
\]
(c) **(24 pts)** If $f$ is continuous on $[a, b]$, prove that

$$
\int_{a}^{b} f(x) \, dx = f(e)(b - a)
$$

for some $e \in [a, b]$.

If you use any major theorems, you must cite them.

4. **(20 pts)** Prove that if the subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ converge to $L$, then the sequence $\{a_{n}\}$ converges to $L$ as well.

5. **(a) (15 pts)** The statement below is an incorrect statement of the Riemann condition:

A function $f$ defined on $[a, b]$ is integrable on $[a, b]$ \hspace{1cm} (1)

$$
\iff
$$

There exists $\epsilon \in \mathbb{R}^+$ such that for all step functions $s$ and $t$

with $s < f < t$ on $[a, b]$, we have $\int_{a}^{b} t - \int_{a}^{b} s < \epsilon$.

This statement says that $(1) \implies (2)$ and $(2) \implies (1)$. Neither of these statements is correct! Prove that $(1) \not\implies (2)$.

(b) **(5 pts)** Give the correct statement of the Riemann condition.

6. **(15 pts)** Suppose that $A$ and $B$ are nonempty subsets of $\mathbb{R}$ such that for all $a \in A$ and $b \in B$, $a \leq b$.

(a) Prove that $\sup A \leq b$, for all $b \in B$.

(b) Prove that $\sup A \leq \inf B$.

7. **(15 pts)** Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$

8. **(15 pts)** If $f$ is a function such that $\lim_{x \to \infty} f(x)$ exists, then

$$
\lim_{x \to a} e^{f(x)} = \exp(\lim_{x \to a} f(x)).
$$

(This follows directly from the fact that the exponential function is continuous.) Use this fact to show that

$$
\lim_{n \to \infty} n^{1/n} = 1.
$$

Do not take the natural log of both sides.

9. **(14 pts)** Determine the radius of convergence $r$ of the following power series:

(a) \( \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} z^{2n}}{2n} \)

(b) \( \sum_{n=1}^{\infty} \frac{(n!)^2 z^n}{(2n)!} \)

10. **(21 pts)** Test the following series for convergence or divergence and give a reason for your decision in each case.
11. (45 pts) Evaluate the integrals below:

(a) $\int x \sin x \cos x \, dx \text{ [hint: use a trig formula.]}$

(b) $\int \frac{(x + 1) \, dx}{(x^2 + 2x + 2)^3}$ [Partial fractions is not necessarily the best method.]

(c) $\int \frac{x + 1}{(x^2 + 1)(x^2)}$ [Just compute partial fracs decomp; do NOT evaluate the integral.]

(d) $\int_0^x |t| \, dt$ (For full credit, figure out how to write your final answer in terms of the absolute value, rather than by cases.)

12. (15 pts) Find the function that the power series below converges to and find the interval of convergence. Justify your answer.

\[ P(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(n - 1)!} \]

13. (10 pts) Compute the limit $\lim_{x \to a^+} \frac{\sqrt{x} - \sqrt{a} + \sqrt{x - a}}{\sqrt{x^2 - a^2}}$. 

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