The Strongest Facets of the Acyclic Subgraph Polytope Are Unknown

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Abstract. We consider the acyclic subgraph polytope and define the notion of strength of a relaxation as the maximum improvement obtained by using this relaxation instead of the most trivial relaxation of the problem. We show that the strength of a relaxation is the maximum of the strengths of the relaxations obtained by simply adding to the trivial relaxation each valid inequality separately. We also derive from the probabilistic method that the maximum strength of any inequality is 2. We then consider all (or almost all) the known valid inequalities for the polytope and compute their strength. The surprising observation is that their strength is at most slightly more than 3/2, implying that the strongest inequalities are yet unknown. We then consider a pseudo-random construction due to Alon and Spencer based on quadratic residues to obtain new facet-defining inequalities for the polytope. These are also facet-defining for the linear ordering polytope.

1 Introduction

Given weights w_a on the arcs of a complete directed graph (or digraph) $D = (V, A)$, the *acyclic subgraph problem* is that of determining a set of arcs of maximum total weight that define an acyclic subgraph. The complement of an acyclic subgraph is called a *feedback arc set*. For general graphs, the acyclic subgraph problem is NP-hard, even for graphs with unit weights and with total indegree and outdegree of every vertex no more than three [GJ79], although the problem is polynomially solvable for planar graphs as was shown by Lucchesi and Younger [LY78]. For any number n of vertices, the acyclic subgraph polytope P^n_{AC} is defined as the convex hull of incidence vectors of acyclic subgraphs of the complete digraph on n vertices; for simplicity, we will omit the superscript n. The acyclic subgraph polytope P_{AC} was extensively studied by Grötschel, Jünger and Reinelt [GJR85a, Ju85, Re85]. At the present time, many classes of facet-defining valid inequalities are known (in addition to the references just cited,

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see, e.g., [Gi90, Ko95, LL94, Su92]; for a survey of many of these inequalities, see Fishburn [Fi92]).

Goemans [Go95] introduced a notion for evaluating the *strength* of a linear programming relaxation for a combinatorial problem, relative to a weaker relaxation of that problem. He applied these results to compute the relative strength of classes of facet-defining inequalities for the traveling salesman problem. Motivated by that definition, we set out to determine the strengths of the known classes of facet-defining inequalities for the maximum acyclic subgraph problem, using as our relaxation for comparison the completely trivial relaxation of P_{AC} given by

$$P = \{x : x_{ij} + x_{ji} \leq 1 \text{ for all } i, j \in V\}.$$

In general, we were aware that certain probabilistic results imply that the strength of P_{AC} itself relative to P must be close to 2 (specifically, $2 - o(1)^5$), something that follows from the fact that, for large random tournaments, the maximum acyclic subgraph has at most about half the total number of arcs (with high probability). Thus we were quite surprised to discover, in computing the strengths of the known inequalities, that in every case except one, their strength was at most 3/2 (see Table 1). In the last case, the strength was still no more than 55/36. This "gap" implies, in particular, that if we were to choose a large random graph and optimize a unit-weight function over the polytope consisting of *all* known valid inequalities, the relaxed solution value would (with high probability) be off by at least 30 percent from the true optimum!

Inequality Type	Reference	General Strength	Max Strength
k-dicycle $(k \geq 3)$	[GJR85a, Ju85]	$k/(k-1)$	$3/2$
k-fence $(k \geq 3)$	[GJR85a, Ju85]	$k^2/(k^2-k+1)$	$9/7 = 1.2857...$
augmented k-fence $(k \geq 3)$	[LL94, Mc90]	$(3k^2-4k)/(2k^2-3k+1)$	$55/36 = 1.5277...$
r-reinforced k-fence $(k \geq 3, 1 \leq r \leq k-2)$	[LL94, Su92]	$\dfrac{k^2+(r-1)k}{k^2-k+(r^2+r)/2}$	$\dfrac{1+\sqrt{3}}{2} = 1.3660...$
k-wheel $(k \geq 3)$	[Ju85]	$10k/(7k-1)$	$3/2$
Z_k $(k \geq 4)$	[Re85]	$(4k+3)/(3k+2)$	$19/14 = 1.3571...$
diagonal	[Gi90]		$< 3/2$
α-critical fence	[Ko95]	$(\|V\|+2\|E\|)/(\alpha(G)+2\|E\|)$	$9/7 = 1.2857...$
node-disjoint k-Möbius ladder $(k \geq 3)$	[GJR85a, Ju85]	$\leq 6k/(5k-1)$	$9/7$

Table 1. Strength of various classes of valid inequalities for P_{AC}.

[5] $o(1)$ means that this term is nonnegative and tends to 0 as the number of vertices n tends to ∞.

Armed with these results, we started hunting for new facet-defining inequalities with strength closer to 2. The probabilistic proof that the strength of P_{AC} is $2-o(1)$ shows that an inequality based on a uniformly selected random tournament has strength $2-o(1)$ with high probability. Many results in random graph theory of this flavor, however, are highly "existential" in nature and indicate no way of explicitly constructing a graph with the desired (highly probable) property. However, in some cases, explicit constructions (often based on number-theoretic arguments) are known which exhibit almost the same properties as their random counterparts. For the maximum acyclic subgraph problem, Alon and Spencer [AS92] explicitly construct such *pseudo-random* tournaments with an upper bound on the size of the maximum acyclic subgraph asymptotically close to one-half. These tournaments are known as *Paley* tournaments.

Asymptotically, these tournaments induce valid inequalities which have strength arbitrarily close to 2, even though there is no guarantee that they define facets of P_{AC}. We have considered Paley tournaments on several small numbers of vertices, and have discovered that the associated Paley inequality on 11 vertices already has strength larger than that of all known valid inequalities, and also defines a facet for P_{AC}. The same results hold for 19 vertices.

The polytope P_{AC} is a cousin of the *linear ordering* polytope P_{LO} (see [GJR85b] and references therein). A linear ordering is a permutation π on the vertex set $\{1, 2, \cdots, n\}$ and will be denoted by $\langle \pi(1), \pi(2), \cdots, \pi(n) \rangle$. Any linear ordering induces an acyclic subgraph $\{(\pi(i), \pi(j)) : i < j\}$. The linear ordering polytope P_{LO} is the convex hull of the maximal acyclic subgraphs of complete graphs induced by linear orderings. Clearly, P_{LO} is a face of P_{AC} [GJR85b]. More precisely, $P_{LO} = \{x \in P_{AC} : x_{ij} + x_{ji} = 1\}$. Moreover, for any nonnegative weight function, any optimal solution over P_{LO} is also an optimal solution over P_{AC}. For technical reasons, our results about the strengths of inequalities will apply only to the acyclic subgraph polytope; nonetheless, our new facet-defining inequalities turn out to be facet-defining for the linear ordering polytope, as well.

The extended abstract is structured as follows. In the next section, we generalize the results on the strength of relaxations derived in [Go95] to polytopes of anti-blocking type, such as P_{AC}. In Section 3, we establish that the strength of P_{AC} is $2-o(1)$, and we compute the strength of almost all known inequalities for P_{AC} in Section 4. In Section 5, we present the Paley inequalities and give a general condition under which they are facet-defining. Finally, in Section 6 we consider a different notion of strength by defining it relative to a stronger LP relaxation of P_{AC}.

2 Strength

A notion of *strength* of a relaxation was introduced by Goemans [Go95] for polyhedra of blocking type. The strength of a relaxation is one measure of how well a relaxation approximates a polyhedron in comparison to another weaker relaxation. In this section, we derive the equivalent results for polytopes of anti-blocking type, i.e., for polyhedra $P \subseteq \mathbb{R}^n_+$ such that $y \in P$ and $0 \leq x \leq y$

imply that $x \in P$. The exposition is adapted from [Go95]. We state the main result (Theorem 2) in the case of polytopes restricted to the nonnegative orthant, although the result holds for more general anti-blocking polyhedra as well.

If P and Q are polyhedra in $\mathbb{R}^n$ then we say that P is a *relaxation* of Q or Q is a *strengthening* of P if $P \supseteq Q$. For polytopes P and Q of anti-blocking type, we say that P is an α-relaxation of Q ($\alpha \in \mathbb{R}, \alpha \geq 1$) or Q is an α-strengthening of P if $Q \supseteq P/\alpha = \{x/\alpha : x \in P\}$, i.e., Q is a relaxation of P/α. Any α-relaxation is also a β-relaxation for any $\beta \geq \alpha$. Also, let $t(P,Q)$ denote the minimum value of α such that P is an α-relaxation of Q. Notice that $t(P,Q) \geq 1$, $t(P,Q) = 1$ if and only if $P = Q$, and that $t(P,Q)$ could be infinite.

The following lemma follows trivially from the separating hyperplane theorem.

Lemma 1. *Let P be a relaxation of a polytope Q, P and Q being of anti-blocking type. Then P is an α-relaxation of Q if and only if, for any nonnegative vector $w \in \mathbb{R}^n$,*

$$Max\{wx : x \in Q\} \geq \frac{1}{\alpha} Max\{wx : x \in P\}.$$

As a corollary, $t(P,Q)$ is equal to

$$t(P,Q) = \mathrm{Sup}_{w \in \mathbb{R}^n_+} \frac{\mathrm{Max}\{wx : x \in P\}}{\mathrm{Max}\{wx : x \in Q\}}, \tag{1}$$

where, by convention, $\frac{0}{0} = 1$.

The following result gives an alternative characterization of $t(P,Q)$ when a description of Q in terms of linear inequalities is known. The proof is similar to that of Theorem 2 in [Go95].

Theorem 2. *Let P and Q be polytopes of anti-blocking type, and let P be a relaxation of Q. Assume $Q = \{x : a_i x \leq b_i \text{ for } i = 1, \ldots, m, x \geq 0\}$, $a_i, b_i \geq 0$ for $i = 1, \ldots, m$. Then*

$$t(P,Q) = Max_i \frac{d_i}{b_i},$$

where $d_i = Max\{a_i x : x \in P\}$.

Proof. From (1), it is clear that

$$t(P,Q) \geq \mathrm{Max}_i \frac{\mathrm{Max}\{a_i x : x \in P\}}{\mathrm{Max}\{a_i x : x \in Q\}} \geq \mathrm{Max}_i \frac{d_i}{b_i}.$$

We therefore need to prove the reverse inequality.

Let w be any nonnegative weight function. By strong duality, we know that

$$\begin{array}{llll} \mathrm{Max}\ wx & & = \mathrm{Min}\ b^T y & \\ s.t. & Ax \leq b & s.t. & A^T y \geq w^T \\ & x \geq 0 & & y \geq 0, \end{array}$$

where ^{T} denotes the transpose. Let y^* be the optimal dual solution of the above program. Then

$$\begin{array}{l}\text{Max } wx \\ s.t.\ x \in P\end{array} \leq \begin{array}{l}\text{Max } (y^*)^T Ax \\ s.t\ \ x \in P\end{array} \leq \sum_i \left\{\begin{array}{l}\text{Max } a_i x \\ s.t.\ x \in P\end{array}\right\} y_i^* = \sum_i d_i y_i^*$$

(the middle inequality follows from the fact that $x \geq 0$ and $(y^*)^T A \geq w^T$). Hence,

$$\frac{\text{Max}\{wx : x \in P\}}{\text{Max}\{wx : x \in Q\}} \leq \frac{\sum_i d_i y_i^*}{\sum_i b_i y_i^*} = \sum_i \left(\frac{b_i y_i^*}{\sum_j b_j y_j^*}\right) \frac{d_i}{b_i}.$$

Since $d_i \geq 0$ (since $a_i \geq 0$) and $y_i^* \geq 0$, the latter quantity can be interpreted as a convex combination of $\frac{d_i}{b_i}$ and is therefore less than or equal to

$$\text{Max}_i \frac{d_i}{b_i}.$$

The result is proved by taking the supremum over all nonnegative weight functions w.

Theorem 2 can be rephrased as follows. To compute $t(P,Q)$, one only needs to consider the cases in which a single inequality of Q is added to P. This motivates the following definition. The *strength* of an inequality $ax \leq b$ with respect to a polytope P is defined as

$$\frac{\text{Max}\{ax : x \in P\}}{b}.$$

Theorem 2 implies that $t(P,Q)$ is equal to the maximum strength with respect to P of a facet-defining inequality for Q.

3 Strength of the Acyclic Subgraph Polytope

The acyclic subgraph polytope P_{AC} is of anti-blocking type, and therefore the results of the previous section apply.

Theorem 3. *The strength $t(P, P_{AC})$ of the acyclic subgraph polytope P_{AC} on n vertices is $2 - O(1/\sqrt{n})$.*

Proof. We first show that the strength of P_{AC} is at most 2. This is a well-known result. Given any nonnegative weight function w, consider the acyclic subgraphs induced by two opposite linear orderings, i.e. $\langle 1, 2, \cdots, n\rangle$ and $\langle n, n-1, \cdots, 2, 1\rangle$. Clearly, every arc is in one of these linear orderings and, therefore, the maximum weight of the acyclic subgraphs induced by these 2 orderings is at least $\frac{1}{2}\sum_{i,j} w_{ij}$. Thus, using (1), $t(P, P_{AC})$ is at most

$$t(P, P_{AC}) \leq \text{Sup}_{w \in \mathbb{R}_+^n} \frac{\sum_{i<j} \max(w_{ij}, w_{ji})}{\frac{1}{2}\sum_{i<j}(w_{ij} + w_{ji})} \leq 2.$$

To establish a lower bound on $t(P, P_{AC})$, we use a probabilistic result. A *tournament* is a directed graph $D = (V, A)$ in which for every (i, j) exactly one of (i, j) or (j, i) belongs to A. We can associate a weight function to every tournament: $w_{ij} = 1$ if $(i, j) \in A$ and 0 otherwise. The strength $t(P, P_{AC})$ given in (1) must be at least the maximum ratio obtained by considering only the weight functions associated with tournaments. Consider now a random tournament chosen uniformly among all tournaments. Erdös and Moon [EM65] have shown that the size $f(n)$ of the maximum acyclic subgraph in a random tournament is $f(n) = \frac{1}{2}\binom{n}{2} + O(n^{3/2}\sqrt{\ln n})$ with high probability and this was refined by Spencer [Sp71] (and de la Vega [Ve83]) to $f(n) = \frac{1}{2}\binom{n}{2} + \Theta(n^{3/2})$ with high probability. As a result, a random tournament gives a ratio of $\frac{\binom{n}{2}}{f(n)} = 2 - O(1/\sqrt{n})$ with high probability, and the worst tournament must give a ratio at least as high.

4 Strength of Known Valid Inequalities

We begin by characterizing certain properties of valid inequalities for P_{AC}; these facts can be found in Jünger [Ju85]. Since P_{AC} is of anti-blocking type, all its facet-defining valid inequalities, except the nonnegativity constraints, are of the form $ax \leq b$ with $a \geq 0$ and $b > 0$. Moreover, any facet-defining valid inequality $ax \leq b$, except the nonnegativity constraints and the inequalities $x_{ij} + x_{ji} \leq 1$, must satisfy $\min(a_{ij}, a_{ji}) = 0$ for all (i, j). We can therefore restrict our attention to such *support reduced* inequalities [BP91].

For the acyclic subgraph problem and the trivial relaxation P given in Section 1, it is very easy to compute the strength of any support reduced inequality $ax \leq b$ with $a, b \geq 0$. Indeed, when optimizing over P, the problem decomposes over all pairs of indices and, as a result, the strength of $ax \leq b$ is given by

$$\frac{\sum_{i<j} \max(a_{ij}, a_{ji})}{b} = \frac{\sum_{i,j} a_{ij}}{b}. \tag{2}$$

In particular, if the inequality is a *rank* inequality, i.e. an inequality of the form $x(A) \leq b$ where $x(A) = \sum_{(i,j)\in A} x_{ij}$, as are most known facet-defining valid inequalities for P_{AC}, then the strength is simply $|A|/b$.

We can use these formulas to compute the strength of most classes of known facet-defining valid inequalities for P_{AC}. However, because of space limitations, we do not include the description of the known inequalities. We refer the reader to Fishburn [Fi92] for a survey. We summarize our findings in Table 1. All the inequalities listed in the table are rank inequalities except the augmented k-fence, the r-reinforced k-fence, and the diagonal inequalities. The column "Max Strength" indicates the maximum value over all possible values of the parameters of an inequality in the class considered. As mentioned before, the strength of any of these inequalities appears to be at most 3/2, except for the augmented k-fences, whose strength is maximized for $k = 5$ at 55/36.

We would like to comment on three of the entries. The node-disjoint k-Möbius ladder inequalities are rank inequalities $x(A) \leq b$ with $b = |A| - \frac{k+1}{2}$. The value of the strength indicated in the table follows from the fact that $|A|$ can be seen to be at least $3k$ when the cycles in the definition of the Möbius ladder are node-disjoint. For the diagonal inequalities, we have not computed their exact strength, but we use the following result.

Lemma 4. *Let $ax \leq b$ be a support reduced valid inequality for P_{AC} and assume that $ax \leq b$ is implied (over P_{LO}) by the valid inequalities $c_i x \leq d_i$ for P_{AC} and the equalities $x_{ij} + x_{ji} = 1$ for all i, j valid only over P_{LO}. Then the strength of $ax \leq b$ is at most the maximum strength of the inequalities $c_i x \leq d_i$. Moreover, the strength is strictly less than the maximum strength if the equalities $x_{ij} + x_{ji} = 1$ are needed.*

Proof. Let l_i/d_i be the strength of inequality $c_i x \leq d_i$, where l_i denotes the sum of the entries of c_i. Since $ax \leq b$ is implied by $c_i x \leq d_i$ and the equalities $x_{ij} + x_{ji} = 1$, there must exist $\lambda_i \geq 0$ and μ_{ij} such that $b = \sum_i \lambda_i d_i - \sum_{ij} \mu_{ij}$ and $a = \sum_i \lambda_i c_i - \sum_{ij} \mu_{ij} e_{ij}$, where e_{ij} is the incidence vector of $x_{ij} + x_{ji}$. Since $ax \leq b$ is support reduced, all μ_{ij}'s must be nonnegative.

The strength of $ax \leq b$ is therefore

$$\frac{\sum_i \lambda_i l_i - 2\sum_{ij} \mu_{ij}}{\sum_i \lambda_i d_i - \sum_{ij} \mu_{ij}} \leq \frac{\sum_i \lambda_i l_i}{\sum_i \lambda_i d_i} \leq \max_i \frac{l_i}{d_i},$$

the first inequality following from the fact that the strength of any inequality is less than 2. Moreover, the first inequality is strict if any $\mu_{ij} > 0$.

As an illustration, the lemma shows that the strength of the k-dicycle inequalities ($k > 3$) is less than the strength of the 3-dicycle inequalities since the k-dicycle inequalities are implied by the 3-dicycle inequalities over P_{LO} (but not over P_{AC}). For the diagonal inequalities, Leung and Lee [LL94] show that they are implied by the 3-dicycle inequalities, the r-reinforced k-fence inequalities and the equalities $x_{ij} + x_{ji} = 1$. This in conjunction with the lemma shows that the strength of the diagonal inequalities is less than $3/2$.

The α-critical fence inequalities [Ko95] are generalizations of k-fence inequalities in which the structure of the inequality's supporting digraph is related to a connected undirected graph $G = (V, E)$ with $|V| \geq 3$. Koppen proves that the associated inequality is a facet if and only if the graph is α-critical, i.e., if any edge is removed from G then the independence number of G, $\alpha(G)$, increases. For this inequality, $\sum_{i,j} a_{ij}$ is $|V| + 2|E|$, while $b = \alpha(G) + 2|E|$, implying that their strength is $\frac{|V|+2|E|}{\alpha(G)+2|E|}$. We show next that the strength of these inequalities is bounded by $9/7$.

Lemma 5. *For any connected, α-critical graph $G = (V, E)$ with $|V| \geq 3$,*

$$\frac{|V| + 2|E|}{\alpha(G) + 2|E|} \leq \frac{9}{7},$$

and this bound is attained if G is a triangle.

Proof. For simplicity, let $v = |V|$, $e = |E|$, and $\alpha = \alpha(G)$. Since G is α-critical and $|V| \geq 3$, it cannot be a tree [LP86, Th. 12.1.8.], and therefore $e \geq v$.

From Turán's theorem [Tu41], we have that $\alpha \geq \frac{v}{d+1}$ where d is the average degree. Thus $\alpha \geq \frac{v}{1+2e/v}$. Letting $x = e/v \geq 1$, the bound on the strength becomes

$$\frac{v+2e}{\alpha+2e} \leq \frac{v+2e}{\frac{v^2}{2e+v}+2e} = \frac{(2e+v)^2}{v^2+4e^2+2ev} = \frac{1+4x+4x^2}{1+2x+4x^2} \leq \frac{9}{7},$$

the value of $9/7$ being attained at $x = 1$. When $x = 1$, G must be a cycle; but, among all cycles, Turán's theorem is tight only for the triangle.

Given a valid inequality for P_{AC}, several operations are known for deriving other valid inequalities [Ju85]. For example, both the operations of node-splitting and arc-subdivision take an inequality $ax \leq b$ and transform it into another inequality $cx \leq d$ with the property that $d - b = \sum c_{ij} - \sum a_{ij} \geq 0$. Such a transformation results in an inequality of lesser strength since both the numerator and denominator of $\sum a_{ij}/b$ increase by the same amount.

Finally, we note that the class of Möbius ladder inequalities introduced by Grötschel, Jünger, and Reinelt [Ju85, GJR85a] is much more general than what we have described here. It is quite possible that there exist general Möbius inequalities whose strength is greater than 3/2. However, the description of these general inequalities gives no systematic way of recognizing classes for which the strength might be greater. (Indeed, our new inequalities, described in the next section, could in fact be Möbius ladders; we comment further about this possibility at the end of the next section.) Also, we have omitted the web inequalities [Ju85, GJR85a] because they are defined relative to general Möbius ladders. When they are defined relative to node-disjoint Möbius ladders, their strength is at most $(6k+10)/(5k+9) < 6/5$.

5 Paley Inequalities

Motivated by the results of Section 3 and 4, we introduce a class of valid inequalities based on a construction of Alon and Spencer [AS92] and we give conditions under which these inequalities define a facet of the linear ordering and acyclic subgraph polytopes.

5.1 Number-Theoretic Preliminaries

We collect here some basic number theoretic results which will be useful for the definition and properties of Paley inequalities. The results can be found in Hardy and Wright [HW79], for example.

The letter p will always denote a prime number. Given p, any x not congruent to 0 modulo p has a unique inverse modulo p, denoted by x^{-1}, i.e. $xx^{-1} \equiv 1 \pmod{p}$. We can thus define a division modulo p by $x/y = xy^{-1}$.

A given $a \not\equiv 0 \pmod p$ is called a *quadratic residue* of p if the congruence $x^2 \equiv a \pmod p$ has a solution x; otherwise a is called a *quadratic non-residue.* There are exactly $(p-1)/2$ quadratic residues of an odd prime p. If we let $\chi(a)$ be 1 if a is a quadratic residue, 0 if a is 0, and -1 if a is a quadratic non-residue, then it is known that $\chi(a) \equiv a^{(p-1)/2} \pmod p$ [HW79, Th. 83]. Since $\chi(ab) = \chi(a)\chi(b)$, the product of two quadratic residues (or two quadratic non-residues) is a quadratic residue, and the product of a quadratic residue and a quadratic non-residue is a quadratic non-residue. For primes of the form $4k+3$, exactly one of a or $-a$ is a quadratic residue [HW79, Th. 82]. For any odd prime p, there exist quadratic residues a, b and c such that $a + b + c \equiv 0 \pmod p$ [HW79, Th. 87].

The number of positive integers not greater than and prime to m is denoted by $\phi(m)$. The function ϕ is called *Euler's function.* For a prime p, $\phi(p) = p - 1$. In general, if m is expressed in its standard form $m = p_1^{a_1} p_2^{a_2} \cdots p_l^{a_l}$ where the p_i's are distinct primes then $\phi(m) = m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_l}\right)$ [HW79, Th. 62]. The *order* of an element $a \pmod m$ is the smallest positive value d such that $a^d \equiv 1 \pmod m$. If a is prime to m then the order of $a \pmod m$ divides $\phi(m)$ [HW79, Th. 88]. The integer a is called a *primitive root* of m if its order is equal to $\phi(m)$. It is known that every prime p has exactly $\phi(p-1)$ primitive roots [HW79, Th. 110]. We will refer to the square of a primitive root as a *squared primitive root.* Observe that any squared primitive root is a quadratic residue with order exactly $(p-1)/2$. If a is a primitive root of a prime p then $\chi(a) \equiv a^{(p-1)/2} \equiv -1 \pmod p$, i.e. a is a quadratic non-residue. Therefore, if p is of the form $4k+3$, then $-a$ is a quadratic residue and cannot be a primitive root. This implies that for a prime p of the form $4k + 3$ the squares of the primitive roots are distinct, and thus there are $\phi(p-1) = \phi((p-1)/2)$ squared primitive roots. In particular, if $(p-1)/2$ is prime, all quadratic residues, except the residue 1, are squared primitive roots.

5.2 Paley tournaments

A *Paley tournament* can be constructed as follows. Take a prime p of the form $4k + 3$. Let the vertex set V be the residues modulo p or $\{0, 1, \cdots, p-1\}$, and let the arc set A be $\{(i,j) : \chi(j-i) = 1\}$. The fact that $\chi(a) = -\chi(-a)$ implies that (V, A) is a tournament. Let $s(\pi)$ denote the number of arcs in the acyclic subgraph defined by a linear ordering π on V, and let $l(p)$ be the maximum of $s(\pi)$ over all π. Alon and Spencer [AS92] show that $l(p) \leq \frac{1}{2}\binom{p}{2} + O(p^{3/2}\log p)$. Given a Paley tournament (V, A), we can associate the inequality

$$\sum_{(i,j)\in A} x_{ij} \leq l(p), \tag{3}$$

which is valid for both P_{AC} and P_{LO}. We refer to this inequality as a *Paley inequality.* The term Paley comes from R.E.A.C. Paley [Pa33] who introduced a very closely related construction for a Hadamard matrix of size $p + 1$.

5.3 Facets

In this section, we give conditions under which the Paley inequality defines a facet of either P_{LO} or P_{AC}.

We first consider the case $p = 7$. The quadratic residues modulo 7 are 1, 2 and 4. We claim that, for $p = 7$, the Paley inequality is implied by the 3-dicycle inequalities. Indeed, if we consider the seven dicycle inequalities corresponding to vertices i, $i-1$, and $i-4$, $i = 0, \ldots, 6$, where all differences are modulo 7, then we cover each arc of the Paley tournament exactly once, and so we have $\sum_{(i,j)\in A} x_{ij} \leq 14$. This value is attainable, e.g., by the permutation $\langle 0, 1, 2, 3, 4, 5, 6\rangle$.

For higher values of p, we need the following definition. Given a linear ordering π, we say that π *contains the loose triple* (u, v, w), where u, v, w are (not necessarily all distinct) quadratic residues summing to 0 modulo p, if there is an index i such that $\pi(i+1) - \pi(i) = u$, $\pi(i+2) - \pi(i+1) = v$ and $\pi(i) - \pi(i+2) = w$.

Theorem 6. *For a prime $p \geq 11$ of the form $4k+3$, the inequality (3) defines a facet of P_{LO} if there exists an optimal linear ordering π (i.e. for which $s(\pi) = l(p)$) which contains a loose triple (u, v, w) where one of $v/u, w/v$ or u/w is a squared primitive root.*

Proof. We start by observing that $v/u, w/v$ and u/w cannot all be equal. If they were all equal say to x then $x^3 \equiv 1 \pmod p$, which contradicts the fact that x must be a squared primitive root (unless $p = 7$ which is ruled out by assumption).

We will consider several operations on linear orderings of Paley tournaments which preserve optimality. Given π and a quadratic residue a, we let $a\pi$ denote the permutation defined by $(a\pi)(i) = \pi(i)\cdot a \pmod p$. Similarly, for any residue a (not necessarily quadratic), we let $\pi + a$ denote the permutation defined by $(\pi + a)(i) = \pi(i) + a \pmod p$. Observe that, in both cases, $s(a\pi) = s(\pi)$ and $s(\pi + a) = s(\pi)$. The effect of these operations on a loose triple (u, v, w) of π is the following: (u, v, w) is still a loose triple of $\pi + a$, and (au, av, aw) is a loose triple of $a\pi$. Also, if π has a loose triple, say given by the index i, then we can obtain two other permutations $(\rho_i \circ \pi)$ and $(\rho_i^2 \circ \pi)$ by applying (once or twice) the permutation ρ_i that only rotates elements $\pi(i), \pi(i+1)$ and $\pi(i+2)$. The important observation is that, by definition of a loose triple, we have $s(\pi) = s(\rho_i \circ \pi) = s(\rho_i^2 \circ \pi)$. Moreover, by applying these rotations, we can assume without loss of generality that the loose triple (u, v, w) in some optimal ordering π satisfies: (i) v/u is a squared primitive root, and (ii) $w/v \not\equiv v/u$. Furthermore, by considering $a\pi$, we can assume that $u = 1$, and then by considering $\pi + a$, we can assume that $\pi(i) = 0$. We therefore assume the existence of an optimal linear ordering π with a loose triple $(1, k, kl)$ at index i where k is a squared primitive root, $k \neq l$ and $\pi(i) = 0$.

To prove that (3) defines a facet of P_{LO}, we consider the set $\mathcal{O} = \{\pi : s(\pi) = l(p)\}$ of optimal orderings and show that if the equality $c^T x = b$ is satisfied for all incidence vectors x^π corresponding to permutations π in $\mathcal{O}$ then $c^T x = b$

is implied by the equality version of (3) and the equalities $x_{ij} + x_{ji} = 1$ for all i, j. Because of the latter equalities, we can assume that the only non-zero coefficients of c correspond to arcs in the Paley tournament, and, as a result, we only need to show that such restricted c's are multiples of the equality version of (3). For simplicity of notation, we let $c_{u,a}$ denote the coefficient of the arc $(u, u+a)$ where a is a quadratic residue. The subscripts of c should always be taken modulo p.

Comparing $c^T x^{\pi_1}$ and $c^T x^{\pi_2}$ where $\pi_1 = \rho_i \circ \pi$ and $\pi_2 = \rho_i^2 \circ \pi$, we derive that $c_{0,1} = c_{1,k}$. If we now consider $\pi' = a\pi + b$ where a is a quadratic residue, the same argument shows that $c_{b,a} = c_{a+b,ak}$. In particular, taking $(a, b) = (k, 1)$, we get that $c_{0,1} = c_{1,k} = c_{k+1,k^2}$. Repeating this process, we derive that, for any positive integer j, $c_{0,1} = c_{d(k,j),k^j}$ where $d(k,j) = k^{j-1} + k^{j-2} + \cdots + 1$. Since k was assumed to be a squared primitive root, as j runs from 0 to $(p-1)/2 - 1$, k^j runs over all quadratic residues modulo p.

If we compare now $c^T x^{\pi}$ with $c^T x^{\pi_2}$, we derive that $c_{0,1} = c_{1+k,kl}$. But we already know that $c_{0,1} = c_{d(k,j),kl}$ where j is such that k^j is congruent to the quadratic residue kl modulo p. We claim that $k + 1 \not\equiv d(k,j) \pmod p$. Assuming the claim, we are almost done. We have just proved that $c_{b_1,kl} = c_{b_2,kl}$ for two distinct values b_1, b_2. By repeatedly adding $b_2 - b_1$ to the permutations, we derive that $c_{0,kl} = c_{b,kl}$ for any b. By multiplying by all quadratic residues, we get that $c_{0,a} = c_{b,a}$ for all b and all quadratic residues a. Now using the fact derived in the previous paragraph that, for any quadratic residue, $c_{0,1} = c_{m,a}$ for some m, we get that all coefficients of $c^T x$ are identical, proving the result.

We still need to prove the claim that $k + 1 \not\equiv d(k,j) \pmod p$. To avoid having to deal with the cases $j = 0$ or $j = 1$ separately, we add $(p-1)/2$ to j in such cases (remember that $k^{(p-1)/2} \equiv 1 \pmod p$), and therefore we assume without loss of generality that $2 \le j \le (p-1)/2 + 1$. We need to show that $d(k,j) - k - 1 \not\equiv 0 \pmod p$. We observe that

$$\begin{aligned} (k-1)(d(k,j) - k - 1) &\equiv (k-1)(k^{j-1} + k^{j-2} + \cdots k^2) \pmod p \\ &\equiv k^j - k^2 \pmod p \\ &\equiv k^2(k^{j-2} - 1) \pmod p \end{aligned}$$

Since k is a squared primitive root, we have that $k \neq 1$ and $k^{j-2} \not\equiv 1 \pmod p$ unless $j = 2$. However, our assumption that $k \neq l$ implies that $j \neq 2$. Therefore, $d(k,j) - k - 1 \not\equiv 0 \pmod p$, proving the claim.

Since the inequality (3) is support reduced, if it defines a facet of P_{LO} then it also defines a facet of P_{AC}. (See Boyd and Pulleyblank [BP91] and Leung and Lee [LL94] for the details of why generally such an implication holds.)

Corollary 7. *Under the same conditions as in Theorem 6, inequality (3) defines a facet of P_{AC}.*

For $p = 11$, 19, and 23, we have been able to compute the correct right-hand side value $l(p)$ using CPLEX branch-and-bound code; see Table 2. In the case

$p = 11$, the following permutation, which yields a solution containing 35 arcs, satisfies the conditions of the proof:

$$\langle 1, 6, 7, 10, 4, 8, 2, 0, 5, 9, 3 \rangle.$$

Notice that the first three elements induce a loose triple $(5, 1, 5)$. Since 5 is a squared primitive root, the theorem can be applied. For $p = 19$, we have the following permutation containing 107 arcs:

$$\langle 8, 17, 18, 15, 5, 3, 12, 10, 2, 0, 9, 7, 16, 14, 6, 4, 13, 11, 1 \rangle.$$

The first three elements induce the loose triple $(9, 1, 9)$. Since 9 is a squared primitive root for 19, we again have the necessary conditions that imply the constraint is facet-defining.

Since the Paley inequalities can be trivially lifted to larger instances, we therefore have the following corollary to Theorem 6 and Corollary 7.

Corollary 8. *The Paley inequalities on 11 and on 19 vertices are facet-defining for P^n_{AC} and P^n_{LO}, for all $n \geq 11$ and 19, respectively.*

In the case of $p = 23$, two rather striking things occurred. First, the only permutations we have found that lie on the facet have no loose triples; moreover, the solutions we have found are all isomorphic to the permutation $\langle 0, 1, 2, \cdots, 21, 22 \rangle$. Second, the strength 253/161 of this inequality is *identical* to that of $p = 11$, 55/35. For both of these reasons we conjecture that the Paley inequality in this case is not facet-defining.

For $p = 31$ and $p = 43$, we have been unable to ascertain whether the best feasible solutions generated (with 285 and 543 edges, respectively) are optimal. In the case of $p = 31$, if the true right-hand side is 285, then we are able to show that the inequality is facet-defining. Although the proof of this fact does not follow directly from the previous theorem, it uses a similar technique of combining "loose tuples" (in this case, tuples larger than triples) in order to equate coefficients.

We do not know whether the Paley inequalities are special cases of Möbius ladders. However, we do know that if they are, then, in the case of $p = 11$, the Möbius ladder would have to have 39 cycles, while the $p = 19$ case would require 127 cycles! Finally, we note that the Paley inequalities are but one type of tournament with pseudo-random properties; a larger class that subsumes the Paley inequalities can be extracted from Hadamard matrices. Our experiments with additional inequalities generated from small instances of Hadamard matrices have thus far yielded inequalities that are neither strong nor provably facet-defining.

6 Dicycle strengths

One drawback to the definition of the strength of an inequality is that strength is defined relative to a given relaxation. An inequality which is strong relative

p	$l(p)$	# Edges	Strength
$p = 11$	35	55	1.57142857...
$p = 19$	107	171	1.59813084...
$p = 23$	161	253	1.57142857...
$p = 31$	≥ 285	465	$\leq$ 1.64893617...
$p = 43$	≥ 543	903	$\leq$ 1.66298342...

Table 2. Examples of Paley inequalities for specific values of p.

to one relaxation might actually be much weaker relative to another relaxation. In this section, we consider a stronger and classical relaxation of P_{AC} as our "benchmark". Some, but not all, of the results corroborate the observations from the previous sections.

Let

$$P_d = \{x : x_{ij} + x_{ji} \leq 1 \text{ for all } i, j \in V \text{ and}$$
$$\sum_{(i,j)\in C} x_{ij} \leq |C| - 1 \text{ for all dicycles } C\},$$

namely, P_d is the relaxation obtained by adding all dicycle inequalities to P. We can apply the results of Section 2 to evaluate the strength of any relaxation relative to P_d instead of relative to P. To avoid any confusion, let the *dicycle strength* of any relaxation Q of P_{AC} with $P_d \supseteq Q$ be defined as $t(P_d, Q)$. Because of Theorem 2, we know that the dicycle strength of any relaxation Q is equal to the maximum dicycle strength of (the relaxation obtained by adding to P_d) any inequality defining Q. The computations of dicycle strengths are slightly more tedious than the computations of strengths; we have nevertheless been able to compute the dicycle strengths of most of the inequalities considered before. Table 3 summarizes these results.

By the asymptotic properties of Paley tournaments, we know that the dicycle strength of the Paley inequalities approaches 4/3 as p tends to infinity. However, the convergence is very slow, and in fact the dicycle strength for $p = 11$ or $p = 23$ is only $22/21 = 1.0476...$ while the dicycle strength for $p = 19$ is $114/107 = 1.0654...$. Thus the dicycle strengths of the Paley inequalities for $p = 11, 19$, and 23 are actually smaller than the dicycle strength of (for example) the 4-fence, which is 14/13; these results are in contrast to those concerning the (regular) strengths of these inequalities. For $p = 31$, the dicycle strength would be $62/57 = 1.0877...$ if in fact $l(31) = 285$.

The dicycle strength of P_{AC} itself must be at least 4/3 (asymptotically, as the number of vertices grows to infinity) since the Paley inequalities achieve this bound. However, we do not know if the dicycle strength of P_{AC} is 4/3 or whether it is larger (possibly as large as 2). On the one hand, if the dicycle strength of P_{AC} is more than 4/3, then there must exist inequalities which are stronger

Inequality Type	Dicycle Strength	Max Value
k-fence	$\dfrac{k^2-k/2}{k^2-k+1}$	$14/13 \quad = 1.0769...$
augmented k-fence	$\dfrac{2k^2-5k/2}{2k^2-3k+1}$	$21/20 \quad = 1.05$
r-reinforced k-fence	$\dfrac{k(k-1)+rk/2}{k(k-1)+r(r+1)/2}$	$1/2+\sqrt{6}/4 = 1.1123...$
k-wheel	$7k/(7k-1)$	$21/20 \quad = 1.05$
Z_k	$(6k+5)/(6k+4)$	$29/28 \quad = 1.0357...$
α-critical fence	$\dfrac{\lvert V\rvert/2+2\lvert E\rvert}{\alpha(G)+2\lvert E\rvert}$	$\leq \dfrac{12+7\sqrt{3}}{12+6\sqrt{3}} = 1.0773...$
Paley	$\dfrac{p(p-1)}{3l(p)}$	$4/3 \quad = 1.3333...$

Table 3. Dicycle strength of various classes of valid inequalities for P_{AC}.

than the Paley inequalities in the dicycle-strength sense. On the other hand, if the dicycle strength of P_{AC} were bounded away from 2, say at most $c < 2$, this fact would be extremely interesting from an approximation point-of-view since it would imply that the value obtained by optimizing over P_d (and this can be done in polynomial time) is within a ratio of c of the value of the maximum acyclic subgraph. The problem of finding an approximation algorithm with a performance ratio better than $2-o(1)$ for the directed acyclic subgraph problem has been a long-standing open problem, at least in part due to the poor upper bounds used in proving the performance guarantees.

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