18.455 Advanced Combinatorial Optimization March 2020

Matching Polytope and Totally Dual Integrality

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Based on scribed notes by Debmalya Panigrahi and from past courses.

In this lecture, we will focus on Total Dual Integrality (TDI) and its application on deriving a complete description of the matching polytope in terms of linear inequalities. We will also introduce the notion of a Hilbert basis and point out its connection to TDI.

1 The Matching Polytope

Given an undirected graph $G = (V, E)$, a *matching* $M \subseteq E$ is a subset of edges such that no two edges in M share a common vertex. We can identify M with its incidence vector:

$$\chi(M) \in \mathbb{R}^{|E|} \quad : \quad (\chi(M))_e = \begin{cases} 1 & \text{if } e \in M, \\ 0 & \text{otherwise.} \end{cases}$$

We define the *matching polytope* of G, $P_M = P_M(G)$ to be the convex hull of these incidence vectors, i.e.

$$P_M(G) = conv\{\chi(M) : M \text{ is a matching of } G\}.$$

Note that since the number of matchings in G is finite, $P_M(G)$ is a convex polytope.

Our goal is to represent $\mathcal{P}$ by a set of linear inequalities defined on a set of $|E|$ variables, $\{x_e \in \mathbb{R}\}_{e \in E}$. We must have $x_e \geq 0$, $\forall e \in E$. Also, every vertex can have at most one adjacent edge in any matching, i.e.

$$x(\delta(v)) \triangleq \sum_{e \in \delta(v)} x_e \leq 1,$$

where $\delta(v)$ is the set of edges incident on vertex v. Thus our first attempt at a linear description of P_M is

$$P = \left\{ (x_e \in \mathbb{R})_{e \in E} \ : \ \begin{array}{ll} x_e \geq 0 & \forall e \in E \\ x(\delta(v)) \leq 1 & \forall v \in V \end{array} \right\}.$$

Since P is a convex subset of $\mathbb{R}^{|E|}$ and $\chi(M) \in P$ for each matching M, it follows from the definition of convex hull that $P_M \subseteq P$. However, as illustrated by the following example, $P_M \subsetneq P$ in general since P can have non-integral extreme points. Consider the triangle (K_3)—its matching polytope is

$$P_M = conv\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}.$$

The point $(0.5, 0.5, 0.5) \in P$, i.e. it satisfies the constraints above; however it is not in the convex hull of the matching vectors.

The above example motivates the following family of additional constraints (introduced by Edmonds). Observe that for any matching M, the subgraph induced by M on any odd cardinality vertex subset U has at most $(|U| - 1)/2$ edges. Thus, without losing any of the matchings, we can introduce the following additional constraints:

$$x(E(U)) \triangleq \sum_{e \in E(U)} x_e \leq \frac{|U| - 1}{2}, \qquad U \subseteq V, \ |U| \text{ is odd},$$

where $E(U)$ is the set of edges in the subgraph induced by G on U. These constraints are called the *odd set constraints* or *blossom constraints*. For the triangle, taking $U = V = \{1, 2, 3\}$, we get the constraint $x_1 + x_2 + x_3 \leq 1$. This constraint is violated by the point $(0.5, 0.5, 0.5)$. Thus, our second attempt at a linear description of the matching polytope is

$$Q = \left\{ (x_e \in \mathbb{R})_{e \in E} \ : \ \begin{array}{ll} x_e \geq 0 & \forall e \in E \\ x(\delta(v)) \leq 1 & \forall v \in V \\ x(E(U)) \leq \frac{|U|-1}{2} & \forall U \subseteq V \ : \ |U| \text{ is odd} \end{array} \right\}.$$

The following theorem asserts that this description indeed captures the matching polytope.

Theorem 1 (Edmonds, 1965) *Q is identical to the Matching polytope, i.e. $P_M = Q$.*

Edmonds gave an algorithmic proof for this theorem: for any linear function, he described an algorithm to find an integer solution in Q (ie. a matching) and used strong duality of linear programing to prove that it constitutes an optimum solution when minimizing this linear function over Q. This algorithm is a generalization of the algorithm we saw for finding a maximum cardinality matching in a graph. Instead, we will prove this theorem in a non-algorithmic way over the course of this and the next lecture using the concept of *Total Dual Integrality* (TDI).

2 Total Dual Integrality

Recall the standard formulations of a primal and its dual linear program.

$$(\text{Primal } (P)) \quad \left\{ \begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq b \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{ll} \min & b^\top y \\ \text{s.t.} & A^\top y = c \\ & y \geq 0 \end{array} \right\} \quad (\text{Dual } (D))$$

We define Total Dual Integrality as follows.

Definition 1 (Total Dual Integrality) *A linear system $\{Ax \leq b\}$ (with A and b rational) is Totally Dual Integral (TDI) if for any integral (cost) vector $c \in \mathbb{Z}^n$ for the primal, such that $\max(c^\top x, Ax \leq b)$ is finite (i.e. the primal has a solution), there exists an optimal dual solution $y \in \mathbb{Z}^m$.*

To establish the connection between TDI and Theorem 1, we state the following theorem, which we will prove later on.

Theorem 2 (Edmonds-Giles, 1979) *If a linear system $\{Ax \leq b\}$ is TDI, and b is integral, then $\{Ax \leq b\}$ is integral, i.e. all its extreme points are integral.*

This theorem implies that if we can prove that the linear system Q is TDI (we prove this in the next lecture), then all the extreme points of Q are integral. For rational linear systems, this is equivalent to the polyhedron Q being the convex hull of all integral points contained in it. Hence, this proves Theorem 1.

It is important to note that TDI is not a property of the polyhedron, but of its representation. In fact, the following theorem states that any rational (not necessarily integral) polyhedron has a TDI representation (with a vector b that's not necessarily integral).

Theorem 3 (Edmonds-Giles, 1979) *Let P be a rational polyhedron. Then, $\exists A, b$ such that $P = \{x \ : \ Ax \leq b\}$, $\{Ax \leq b\}$ is TDI and A is integral.*

To illustrate this point, consider the two-dimensional polytope (refer to Figure 1) defined as

$$P = conv\{(0,3), (2,2), (0,0), (3,0)\}.$$

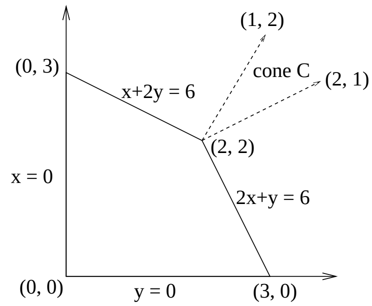

Figure 1: A primal linear system and a dual cone.

This polytope may have many different representations. For example,

$$P = \left\{ \begin{array}{l} x \geq 0, \;\; y \geq 0 \\ x + 2y \leq 6 \\ 2x + y \leq 6 \end{array} \right\}.$$

This linear system, however, is not TDI. For example, if the cost vector is $c^T = (1\ 1)$, then the primal maximum is achieved by $(2, 2)$. However, $(1, 1)$ cannot be expressed as a linear integer combination of $(1, 2)$ and $(2, 1)$, the normals to the tight constraints at $(2, 2)$. Thus, there is no integral dual optimum and P is not TDI.

In Theorem 3, we should emphasize that A is integral, but of course b will only be integral if P itself is integral, see Theorem 2. In the rest of the lecture, we will prove Theorems 2 and 3.

3 Hilbert Basis

We now need to introduce the concept of a *Hilbert basis.*

Definition 2 *A set of vectors* $\{a_1, a_2, \ldots, a_k\}$ *with* $a_i \in \mathbb{Z}^n\ \forall i$ *defines a* Hilbert basis *if for any* $x \in C \cap \mathbb{Z}^n$*, where*

$$C = cone(a_1, a_2, \ldots, a_k) = \left\{ \sum_i \lambda_i a_i : \lambda_i \geq 0,\;\; \lambda_i \in \mathbb{R}\ \forall i \right\},$$

there exists $\mu_1, \mu_2, \ldots, \mu_n$*, such that* $\mu_i \in \mathbb{Z}$*,* $\mu_i \geq 0$ *for each* i *and* $x = \sum_i \mu_i a_i$*.*

The following theorem, then, is a simple consequence of LP duality.

Theorem 4 *A linear system* $\{Ax \leq b\}$ *is TDI iff for each face* F *of* $P = \{x : Ax \leq b\}$*, the normals to the tight constraints for* F *form a Hilbert basis.*

In the above theorem, we could have replaced 'each face' by 'each extreme point', and the proof would also follow easily from LP duality, since for every vector c, there always exists an optimum extreme point.

In our previous example (refer to Figure 1), a Hilbert basis for the cone (the *dual cone* associated with the vertex $(2, 2)$) defined by the vectors $(1, 2)$ and $(2, 1)$ is given by the set of vectors $H = \{(1, 2), (2, 1), (1, 1)\}$. We can get the additional vector $(1, 1)$ by adding the redundant constraint $x_1 + x_2 \leq 4$ in the primal.

In fact, by considering also the dual cones corresponding to the vertices $(3,0)$, $(0,3)$, $(2,2)$ and $(0,0)$, one can show that the linear system

$$\begin{cases} x_1, x_2 & \geq & 0 \\ x_1 + 2x_2 & \leq & 6 \\ 2x_1 + x_2 & \leq & 6 \\ x_1 + x_2 & \leq & 4 \\ x_1 & \leq & 3 \\ x_2 & \leq & 3 \end{cases}$$

is TDI. For example, the cone corresponding to the vertex $(3,0)$ has a Hilbert basis $\{(1,2),(-1,0),(0,1)\}$.

The following theorem, in combination with Theorem 4, proves Theorem 3.

Theorem 5 *Any rational polyhedral*[1] *cone C has a finite integral Hilbert basis.*

Proof: Let $C = \{\sum_i \lambda_i a_i \ : \ \lambda_i \geq 0,\ \lambda_i \in \mathbb{R}\}$, $a_i \in \mathbb{Z}^n$. Define $Q = \{\sum_i \lambda_i a_i \ : \ 0 \leq \lambda_i \leq 1\}$. For any $c \in C \cap \mathbb{Z}^n$,

$$c = \sum_i \lambda_i a_i = \sum_i (\lambda_i - \lfloor \lambda_i \rfloor) a_i + \sum_i \lfloor \lambda_i \rfloor a_i = z + w,$$

where $z = \sum_i (\lambda_i - \lfloor \lambda_i \rfloor) a_i$ and $w = \sum_i \lfloor \lambda_i \rfloor a_i$. Since $a_i \in \mathbb{Z}^n$ and $\lfloor \lambda_i \rfloor \in \mathbb{Z}$ for each i, $w \in \mathbb{Z}^n$. Since $c \in \mathbb{Z}^n$, this implies that $z \in \mathbb{Z}^n$. Clearly, $z \in Q$; hence, $z \in Q \cap \mathbb{Z}^n$. Furthermore, each $a_i \in Q \cap \mathbb{Z}^n$. Hence, c is an integral combination of vectors in $Q \cap \mathbb{Z}^n$. Thus, $Q \cap \mathbb{Z}^n$ is a Hilbert basis for C. $\square$

We now give a proof of Theorem 2.

Proof of Theorem 2: We proceed by contradiction. Consider an extreme point x^* of P such that $x_j^* \notin \mathbb{Z}$ for some j. We can find an integral vector c such that x^* is the unique optimal solution corresponding to c by picking a rational vector c in the interior of the dual cone (always full-dimensional) of x^* and scaling appropriately. Consider $\hat{c} = c + \frac{1}{q} e_j$ where q is an integer. Since the cone is full dimensional, $\hat{c}$ will be in the interior of the dual cone of x^* for a sufficiently large q. Now it follows that $(q\hat{c})^\top x^* - (qc)^\top x^* = x_j^* \notin \mathbb{Z}$. This means that at least one of $(q\hat{c})^\top x^*$ and $(qc)^\top x^*$ is not integral. By duality and the fact that b is integral, we conclude that one of the two corresponding dual optimal solutions (say y and $\hat{y}$) is not integral. This contradicts the TDI property since both qc and $q\hat{c}$ are integral. $\square$

4 Alternative Proof of Integrality for TDI systems

Here we give another proof of Theorem 2 using Kronecker's Approximation Theorem. (This was not covered in lecture.)

4.1 Kronecker's Approximation Theorem

We begin by proving a theorem of Kronecker.

Definition 3 *Let $a_1, \cdots, a_n$ be rational vectors. A* lattice *$L(a_1, a_2, \ldots, a_n)$ is the set $\{\sum_i^n a_i x_i : x_i \in \mathbb{Z}\ \forall i\}$, i.e. is an additive group finitely generated by linearly independent vectors.*

(The assumption that it is finitely generated by *linearly independent* vectors is needed; otherwise, it would not be discrete (consider the 1-dimensional case generated by 1 and $\sqrt{2}$).

[1] i.e. generated by a finite number of vectors

Theorem 6 (Kronecker Approximation Theorem (1884)) *$\exists x \in \mathbb{Z}^n$ s.t. $Ax = b$ if and only if $\forall y, y^\top b$ is an integer whenever $y^\top A$ is an integral vector.*

Proof: To prove the forward implication, take an integral solution x^*. For any y, since $y^\top Ax^* = y^\top b$, if $y^\top A$ is integral then $y^\top b$ must be an integer too.

We next prove the converse. First, suppose that $Ax = b$ does not have a solution (over the reals). Then, there is a solution to $y^\top A = 0$ (integral) with $y^\top b \neq 0$ and, by scaling y appropriately, we can get $y^\top b \notin \mathbb{Z}$. Thus, we can assume that $Ax = b$ has a solution, and by getting rid of redundant equalities, we can assume that A has full row rank.

Let a_j denote the jth column of A. The statement that $\exists x \in \mathbb{Z}^n$ s.t. $Ax = b$ is equivalent to saying that $b \in L(a_1, \ldots, a_n)$, or $L(A)$, generated by the columns of A. We will perform a series of column operations to A resulting in the matrix $A' = [B \;\; 0]$, where B is lower triangular. Each operation will leave the lattice unchanged, i.e. $L(A') = L(A)$, so that $Ax = b$ will have an integral solution iff $A'x = b$ has an integral solution. The operations will also be shown to preserve integrality of $y^\top b$ and $y^\top A$, so that it will suffice to consider A' instead of A in showing the result.

Observe that B must be nonsingular because we have assumed that A has full row rank. Stacking the $y^\top$ to form a matrix, we have that for all matrices Y, if YA is an integral matrix then Yb is an integral vector. Letting $Y = B^{-1}$, note that $B^{-1}A' = \begin{bmatrix} I & 0 \end{bmatrix}$ is integral. Hence, $B^{-1}b$ must be integral. Since

$$A' \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} = b, \tag{1}$$

we have found an integral solution to the system $A'x = b$. All that remains is to (a) describe the operations on the matrix A and show that they preserve integrality, and (b) show how to use them to produce A' from A.

(a) The operations are (i) exchanging two columns, (ii) multiplying a column by -1 and (iii) subtracting an integral multiple of one column from another column. These are unimodular transformations, i.e. $A' = AT$ where $T \in \mathbb{Z}^{n \times n}$ and $\det(T) = \pm 1$, so $T^{-1} \in \mathbb{Z}^{n \times n}$. Thus, any sequence of these operations leaves the lattice intact. Indeed if $Ax = b$ where $x \in \mathbb{Z}^n$ then $A'x' = b$ for $x' = T^{-1}x \in \mathbb{Z}^n$, and vice versa if $A'x' = b$ for $x' \in \mathbb{Z}^n$ then $Ax = b$ for $x = Tx' \in \mathbb{Z}^n$. Also, for any y, $y^T AT$ is an integral vector iff $y^T A$ is an integral vector, so the operations also preserve the property that $y^\top b \in \mathbb{Z}$ whenever $y^\top A \in \mathbb{Z}$.

(b) Using these elementary operations, we can transform A into the form

$$A' = \begin{bmatrix} B & 0 \end{bmatrix} \tag{2}$$

with B lower triangular as follows. For the first row, we can take any two non-zero entries, make them positive by possibly multiplying the column by -1, and compute their gcd using Euclid's algorithm,

$$\gcd(x, y) = \begin{cases} \gcd(x - y, y) & \text{if } x \geq y \\ \gcd(y, x) & \text{if } x < y \\ x, y & \text{if } y = 0 \end{cases} \tag{3}$$

Since these operations are elementary, we can perform them on the columns and reduce the first row to one non-zero entry. We can then put this column as column 1 and proceed to the next row, leaving column 1 fixed. Proceeding in this manner results in the desired form for A'. $\square$

4.2 Proof of Theorem 2

Before showing Theorem 2, we give a corollary of Theorem 6 that we use.

Definition 4 *$H = \{x : c^\top x = \alpha\}$ is a* supporting hyperplane *of the polytope $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ if $H \cap P$ is a non-empty face of P.*

Corollary 7 *$P = \{x \in \mathbb{R}^n : Ax \le b\}$ is integral if and only if every supporting hyperplane of P contains an integral vector.*

Proof: The forward implication is immediate because every supporting hyperplane contains a vertex of P. For the converse, suppose x^* is a non-integral vertex of P. We will demonstrate a hyperplane that does not contain any integral vector. Since x^* is a vertex, there exists a subset $\hat{A}$ of the rows of A (that define P) such that x^* is a unique solution of the system $\hat{A}x = \hat{b}$. Applying Kronecker's approximation theorem, (since x^* is the unique solution and is *not* integral) there must exist a vector $y \in R^m$ such that $y^\top \hat{b}$ is non-integral and $y^\top \hat{A}$ (a vector) *is* integral. Assuming that A and b are rational, we can add an integral constant to the components of y to make y nonnegative while mantaining that $y^\top \hat{b}$ is non-integral and $y^\top \hat{A}$ is integral. Let $c = \hat{A}^\top y$ and $\alpha = y^\top \hat{b}$, and consider $H = \{x : c^\top x = \alpha\}$. Since $\hat{A}x \le \hat{b}$ for all $x \in P$, multiplying both sides by y^T (≥ 0) we obtain that $y^\top \hat{A}x \le y^\top \hat{b}$, i.e. $c^\top x \le \alpha$ for all $x \in P$. The inequalities are tight for x^*, which is in H. We conclude that H is a supporting hyperplane of P. However, c is integral and α is not, so it follows that H cannot contain any integral vector. □

We now finish the proof of Theorem 2.

Proof of Theorem 1: If $Ax \le b$ is TDI and b is integral, pick any integral c. We will show that the supporting hyperplane with normal c contains an integral vector. First, we can assume that the c_j's are relatively prime (otherwise divide by their common gcd). By our TDIness assumption, $\max c^\top x$ subject to $Ax \le b$ will have value an integer α, and $c^\top x = \alpha$ is a supporting hyperplane. We need to show that it contains an integral vector.

Since the entries of c are relatively prime, we can find an integral vector x contained in the supporting hyperplane. (Indeed, it can be shown easily by induction on n that if the gcd of the entries of c is g then there is an integral solution to $c^\top x = g$.) Therefore, we conclude that $Ax \le b$ is integral. □

5 Matchings and the Cunningham-Marsh formula

We now come back to the matching polytope, and show that Edmonds' description using odd cardinality subset constraints is TDI (a result due to Cunningham & Marsh) . Therefore, by Theorem 2, it integral and it provides a complete characterization of the matching polytope in terms of linear inequalities.

Edmonds [2] proved that the matching polytope is given by the following system of linear inequalities:

$$Q = \left\{ (x_e \in \mathbb{R})_{e \in E} \;:\; \begin{array}{ll} \sum_{e \in \delta(v)} x_e \le 1 & \forall v \in V \\ \sum_{e \in E(U)} x_e \le \frac{|U|-1}{2} & \forall U \in \mathcal{P}_{odd} \\ x_e \ge 0 & \forall e \in E \end{array} \right\}.$$

where $\mathcal{P}_{odd} = \{U \subseteq V \;:\; |U| \text{ is odd}\}$ denotes the odd cardinality subsets.

Cunningham and Marsh [1] showed that Q is TDI, providing another proof that Q is the matching polytope, since TDIness implies that all vertices of Q are integral vectors and any valid integer solution of Q is a matching. Consider the dual of $\max_{x \in Q} c^\top x$, where the dual variables are y_v for every vertex $v \in V$ and z_U for every set $U \in \mathcal{P}_{odd}$:

$$\begin{aligned} \min & \sum_{v \in V} y_v + \sum_{U \in \mathcal{P}_{odd}} \frac{|U|-1}{2} z_u \\ \text{s.t.} & \sum_{v \,:\, e \in \delta(v)} y_v + \sum_{U \in \mathcal{P}_{odd} : e \in E(U)} z_U \ge c_e \;\; \forall e \in E \\ & y \ge 0 \end{aligned}$$

TDIness can be stated as follows.

Theorem 8 (Cunningham-Marsh) *For all $c \in \mathbb{Z}^{|E|}$, there exist integral vectors y and z that are feasible and $\sum_{v \in V} y_v + \sum_{U \in \mathcal{P}_{odd}} \frac{|U|-1}{2} z_u \leq \nu(c)$, where $\nu(c)$ is the maximum cost of any matching.*

The proof given here is from Schrijver's book, while Cunningham and Marsh prove it algorithmically.

Proof: We show the result by induction on $|V| + |E| + c(E)$ (recall that $c(E)$ is integral). We can assume that $c(e) \geq 1$ for $e \in E$ (otherwise, delete the edge) and that the graph is connected (otherwise, apply proof to the components). The base case of $|V| = 2, |E| = 1, c(e) \geq 1$ is trivially shown by setting $y_1 = c(e)$ and $y_2 = 0$. Let $D(y, z)$ denote the value of the dual objective.

<u>Case 1</u> $\exists v \in V$ such that every maximum cost matching for c covers v. Define the modified edge costs $c'(e) = c(e)$ for $e \notin \delta(v)$ and $c'(e) = c(e) - 1$ for every $e \in \delta(v)$. By the assumption, the cost of maximum matching $\nu(c')$ is $\nu(c) - 1$. By induction, there exists integral y', z' such that $D(y', z') \leq \nu(c')$. Define $y_v = y'_v + 1$, and $y_u = y'_u$ for $u \neq v$. y and z are feasible since the only constraints changed are for $e \in \delta(v)$ and both $c(e)$ and y_v have increased by 1 above $c'(e)$ and y'_v, respectively. Also, $D(y, z) = D(y', z') + 1 \leq \nu(c') + 1 = \nu(c)$, finishing the induction step.

<u>Case 2</u> Otherwise, $\forall v, \exists$ some maximum cost matching for c that does not cover v. Define the modified edge costs $c'(e) = c(e) - 1 \; \forall e \in E$. We will show that all maximum matchings M for c' miss at least 1 vertex. Let M be a maximum matching for c' with $|M|$ as large as possible. Suppose for contradiction that M covers all vertices. Let N be a maximum cost matching for c that does not cover some vertex. Then,

$$c'(N) = c(N) - |N| > c(N) - |M| \geq c(M) - |M| = c'(M) = \nu(c'),$$

which contradicts the definition of ν (the first inequality is because M covers at least one more vertex than N, and the second inequality is because N is optimal for c).

<u>Case 2a</u> Suppose there exists a maximum cost matching M' for c' such that $|M| = \frac{|V|-1}{2}$, i.e. $|V|$ is odd and M misses precisely one vertex. By induction, there exists integral y', z' such that $D(y', z') \leq \nu(c')$. Let $z = z'$ and $y = y'$, with the exception that $z_V = z'_V + 1$ (for the odd set $V \in \mathcal{P}_{odd}$). Since z_V is included in every constraint and both it and $c(e)$ were increased by one, y, z are feasible. Furthermore, $D(y, z) = D(y', z') + \frac{|V|-1}{2} \leq \nu(c') + \frac{|V|-1}{2} \leq \nu(c)$, where the last inequality is because we can construct a matching for c using the matching for c'. This finishes the induction step for $|M| = \frac{|V|-1}{2}$.

<u>Case 2b</u> Suppose all maximum cost matchings for c' miss at least two vertices. Let M be such a matching with $|M|$ maximum and unmatched vertices u and v closest. Note that u and v cannot be adjacent, i.e. $d(u, v) \geq 2$, since otherwise we could have added this edge to make the matching larger. Let t be the second node on the shortest path from u to v in the graph. Note that t must be matched in M, as otherwise we could increase $|M|$ by matching u and t.

Let N be a maximal matching for c, $c(N) = \nu(c)$, such that t is *unmatched* in N. Look at $M \triangle N$. Since t is matched in M and unmatched in N, $M \triangle N$ has a component P with t as an endpoint. Since every vertex in P has degree at most 2, P must be a path. Let M' be the symmetric difference $M \triangle P$ and $N' = N \triangle P$. Since P is a path containing alternatively edges from M and N, M' and N' are both matchings. Also, $|M'| \leq |M|$ because the last edge of the path (connecting to t) is from M. However,

$$\begin{aligned} c(M) + c(N) &= c(M \triangle P) + c(N \triangle P) \implies & (4)\\ c'(M) + |M| + c(N) &= c'(M \triangle P) + |M \triangle P| + c(N \triangle P) \implies & (5)\\ c'(M) + |M| &\leq c'(M \triangle P) + |M \triangle P|, & (6) \end{aligned}$$

where the last step was because $c(N) = \nu(c) \geq c(N \triangle P)$. However, since $c'(M) = \nu(c') \geq c'(M \triangle P)$ and $|M| \geq |M'|$, Eq. (6) must be an equality and we can conclude that $c'(M') = c'(M) = \nu(c')$ and

$|M| = |M'|$. Note that t is unmatched in M'. Also, P cannot cover *both* u and v, since neither u nor v are covered by M and only one of them (if in N) can be the other endpoint of the path. Thus, $M' = M \triangle P$ does not cover u or v (or both). Suppose it does not cover u. Then, since t is between u and v on the shortest path, we have that $d(u,t) < d(u,v)$. This contradicts our choice of M, u, v since we had assumed that u and v were uncovered vertices with the shortest distance. □

Although we have fully characterized the matching polytope, it has exponentially many constraints. Padberg and Rao [3] give an efficient separation algorithm for the odd cardinality subset constraints that, given x, in polynomial time decides if $x \in Q$. If $x \notin Q$, their algorithm produces a hyperplane that separates x from Q. The separation algorithm works by finding the minimum odd cut in a suitable graph.

In the next few lectures, we consider extended formulations of a polytope P, which is a polytope Q lying in a higher-dimensional space but which projects onto P. Surprisingly, Q may require fewer facets than P. And therefore a very natural question is whether the matching polytope admits an extended formulation with a polynomial (or even subexponential) number of inequalities. This line of work started with the study of Yannakakis [5], and eventually lead to a result of Rothvoss [4] which showed that every extended formulation of the matching polytope has an exponential number of inequalities.

References

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