18.438 Advanced Combinatorial Optimization Michel X. Goemans

Problem Set 2

April 3rd, 2014

This problem set is due in class on April 15th, 2014.

1. Prove that a $0, \pm 1$ matrix which is minimally *not* totally unimodular (i.e. having all square subdeterminants in $\{-1, 0, 1\}$ except for the matrix itself) has determinant ± 2.

   (Hint: think about pivoting on a nonzero element.)

2. Prove that if a matroid is representable over GF(2) and over GF(3) then it can be represented over any field by a totally unimodular matrix (it is thus regular).

   (Hint: Start with a representation $[I|B]$ of the matroid over GF(3) and interpret it as a real matrix with entries $0, \pm 1$. Also use the previous exercise.)

   (Remark: The statement is still true if one replaces GF(3) by any field of charateristic other than 2.)

3. Given a full rank matrix $A \in \mathbb{R}^{n \times n}$ (true also for any field F), let R and C denote the indices of the rows and columns of A. Given $I \subset R$, show using matroid intersection that there exists $J \subset C$ with $|I| = |J|$ such that both $A(I, J)$ and $A(R \setminus I, C \setminus J)$ are of full rank. (Another way of proving this without matroid intersection would be through the generalized Laplace expansion of the determinant.)

4. Suppose we are given an undirected graph $G = (V, E)$, and additional vertex $s \notin V$, an integer k, and we would like to add the minimum number of edges between s and vertices of V (multiple edges are allowed) such that the resulting graph H on $V + s$ has k edge-disjoint paths between any two vertices of V (i.e. the only cut that could possibly have fewer than k edges is the cut separating s from V).

   (a) Argue that this problem is equivalent to finding $x : V \to \mathbb{Z}_+$ minimizing $x(V)$ such that $\forall \emptyset \neq S \subset V$:
   $$x(S) \geq k - d_E(S),$$
   where $d_E(S) = |\delta_E(S)|$ corresponds to the number of edges between S and $V \setminus S$ in G.

   (b) Add k edges between s and each vertex of V. Let A be these $k|V|$ newly added edges. Say that $F \subseteq A$ is feasible if the graph $(V + s, E \cup (A \setminus F))$ has at least k edge-disjoint paths between any two vertices of V. Prove that the feasible sets form the independence sets of a matroid.

   (c) How would you efficiently solve the original problem (with k part of the input)?