

Problem Set 1

March 3rd, 2014

This problem set is due in class on March 13, 2014.

1. Given an *undirected* graph $G = (V, E)$, two specified vertices s and t and cost $c : E \rightarrow \mathbb{R}$, consider the shortest path problem of finding a *simple* (i.e. with all vertices distinct) path of minimum total cost between s and t . In its full generality, if the costs can be negative, the problem is NP-hard (as the Hamiltonian path problem can be reduced to it). If all costs are nonnegative then Dijkstra's algorithm can be used to solve the problem efficiently. But consider the case in which costs may be negative but there are no negative cost cycle. Notice that the trick of bidirecting every edge will not work as this will introduce negative cost cycles (for any negative cost edge).
 - (a) First show that this version of the shortest path problem does *not* necessarily satisfy the optimum subpath problem: If $P = v_0 - v_1 - \dots - v_k$ is a shortest $v_0 - v_k$ path then $v_i - v_{i+1} - \dots - v_{j-1} - v_j$ is a $v_i - v_j$ shortest path.
 - (b) Show that this shortest path problem in G can be reduced to the problem of finding a minimum cost perfect matching in another graph. (See in a few exercises for a converse statement.)
2. Consider the problem of counting the number $\phi(G)$ of perfect matchings in a graph $G = (V, E)$. For any orientation \vec{E} , we can associate a skew-symmetric matrix A_s where

$$A_s(i, j) = \begin{cases} 1 & (i, j) \in \vec{E} \\ -1 & (j, i) \in \vec{E}. \end{cases}$$

Show that

$$\mathbb{E}(\det(A_s)) = \phi(G),$$

when the orientation is chosen uniformly among all $2^{|E|}$ orientations. Deduce that there exists an orientation \vec{E} with

$$\phi(G) \leq \det(A_s) \leq (\phi(G))^2.$$

(However, it is not known how to find such an orientation.)

3. Given a polytope P , two *vertices* (faces of dimension 0) are said to be adjacent if the line segment between them form an *edge* (face of dimension 1). Give a necessary and sufficient condition for the incidence vectors of two perfect matchings M_1 and M_2 to be adjacent on the perfect matching polytope. (Hint: the condition should be in terms of $M_1 \Delta M_2$.)
4. Same question for two stable sets S_1 and S_2 over the stable set polytope. (Your answer to the previous exercise could be deduced from this one by constructing the line graph $L(G)$ of G ; $L(G)$ has a vertex for each edge of E , and two vertices of $L(G)$ are adjacent if the two edges are incident to the same vertex.)
5. Show that if we have an algorithm/oracle for finding an $s - t$ shortest path in an undirected graph with no negative cost cycles then we can use this oracle repeatedly to find in polynomial time a minimum cost perfect matching in a graph. (Hint: use exercise 3. Your algorithm's running time may depend polynomially on the log of the costs (and m and n).)