18.438 Advanced Combinatorial Optimization	March 4, 2014
Lecture 8	
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1 The Weak Perfect Graph Theorem

Given a graph $G = (V, E)$, recall the definitions from last time:

$$\omega(G) = \text{size of largest clique},$$

$$\begin{aligned}\chi(G) &= \text{chromatic number}\\ &= \text{minimum number of colors required to color vertices}\\ &= \text{minimum number of stable sets required to cover vertices},\end{aligned}$$

$$\alpha(G) = \omega(\bar{G}) = \text{size of largest stable set},$$

$$\chi(\bar{G}) = \text{minimum number of cliques required to cover vertices}.$$

For $S \subseteq V$, let $G[S]$ denote the subgraph induced by the vertices in S. Note that $w(G[S]) \leq \chi(G[S])$ because every node in a clique must be colored with a unique color. Perfect graphs are those in which equality holds for every S.

Definition 1 *A graph G is* perfect *if for every $S \subseteq V$, $\omega(G[S]) = \chi(G(S))$.*

Last time we proved the following lemma.

Lemma 1 (Lovász Repetition Lemma) *If G is perfect and G' is obtained from G by "repeating" v then G' is also perfect.*

Recall that "repeating" v means creating a new vertex v' and adding edges $\{(u, v') : (u, v) \in E\}$ and (v, v'). Today we will prove the Weak Perfect Graph Theorem which claims that G is perfect iff its complement $\bar{G}$ is perfect. The proof won't be difficult but illustrates nicely the power of polyhedral approaches in proving combinatorial statements. First define some polytopes:

$$STAB(G) = \text{conv}\,(\mathbf{1}(S) : S \text{ stable in } G) \subseteq \mathbb{R}^V \qquad \text{where } (\mathbf{1}(S))_v = \begin{cases} 1 & v \in S \\ 0 & v \notin S. \end{cases}$$

$$QSTAB(G) = \left\{x \in \mathbb{R}^V : x(C) \leq 1 \ \forall \text{ clique } C,\, x \geq 0\right\}.$$

Note that $STAB(G) \subseteq QSTAB(G)$ because the inequalities that define $QSTAB(G)$ are valid for the indicator vector $\mathbf{1}(S)$ of any stable set S. Now for the main result of this section:

Theorem 2 (Weak Perfect Graph Theorem) *The following are equivalent:*

(i) G is perfect,

(ii) The linear system $\left\{ \begin{array}{l} x(S) \leq 1 \ \forall \textit{ stable set } S \\ x \geq 0 \end{array} \right\}$ *is TDI (totally dual integral),*

(iii) $QSTAB(G)$ is integral, i.e. $QSTAB(G) = STAB(G)$,

(iv) $\bar{G}$ is perfect.

Note that the linear system in (ii) describes $QSTAB(\bar{G})$. This means condition (ii) implies that $QSTAB(\bar{G})$ is integral, i.e. $QSTAB(\bar{G}) = STAB(\bar{G})$.

Proof: (i) $\Rightarrow$ (ii). Consider the primal linear program $\left\{\begin{array}{l} \text{Max } \sum_{v\in V} w_v x_v \\ x(S) \leq 1 \ \forall \text{ stable set } S \\ x \geq 0 \end{array}\right\}$ from (ii) and its dual $\left\{\begin{array}{l} \text{Min } \sum_{S \text{ stable}} y_S \\ \sum_{S\,:\,v\in S} y_S \geq w_v \ \ \forall v \in V \\ y \geq 0 \end{array}\right\}$. We need to show that for every $w \in \mathbb{Z}^V$ there exists an integral optimum solution y for the dual. Apply the repetition lemma iteratively in order to obtain a graph G' that has w_v repeated copies of v for each $v \in V$. Let $\omega(G, w)$ be the weight of a maximum weight clique in G. Note that this is just the maximum integral solution to the primal LP so $\omega(G, w) \leq \sum_{v\in V} w_v x_v^*$ where x^* is a primal optimum solution. Next note that $\omega(G, w) = \omega(G')$ because given a clique of total weight W in G it is easy to produce a corresponding clique of size W in G'; conversely, given a clique of size W in G' you can produce a clique of total weight at least W in G. We also have $w(G') = \chi(G')$ because G' is perfect by the repetition lemma. If we now take the collection of $\chi(G')$ stable sets of G' that cover V' then we can map these back to stable sets of G in the natural way. For each stable set S of G, let y_S be the number of copies of S produced by this "mapping back" procedure. Note that y_S is an integral solution to the dual LP and furthermore must be optimal because there is a primal solution of greater or equal value: $\sum_S y_S = \chi(G') = \omega(G') = \omega(G, w) \leq \sum_{v\in V} w_v x_v^*$.

(ii) $\Rightarrow$ (iii). It is sufficient to let $x \in QSTAB(G) \cap \mathbb{Q}^V$ and show $x \in STAB(G)$. Choose $q \in \mathbb{Z}$ such that $qx \in \mathbb{Z}^V_{\geq 0}$. Define weights $w_v = qx_v \in \mathbb{Z}$ and let C be the maximum weight clique in G. Since $x \in QSTAB(G)$ we have (by the defining inequalities for $QSTAB$) $x(C) \leq 1$. If z is the indicator vector $\mathbf{1}(C)$ then this means $\sum_{v\in V} x_v z_v \leq 1$. We know that the primal LP from (ii) is integral (because it is TDI) and is therefore maximized by z (since z describes the maximum weight clique). The weight of z is $\sum_{v\in V} w_v z_v = \sum_{v\in V} qx_v z_v \leq q$. Therefore the optimum value of the primal from (ii) is at most q. Since it is TDI there must be an integral dual optimum solution y of value $\leq q$. This y must satisfy $\sum y_S \leq q$ and $\sum_{S\,:\,v\in S} y_S \geq w_v \ \ \forall v \in V$. Rewrite these as $\sum \frac{1}{q} y_S \leq 1$ and $\sum_{S\,:\,v\in S} \frac{1}{q} y_S \geq \frac{w_v}{q} = x_v \ \ \forall v \in V$. By modifying the y_S variables appropriately (and no longer requiring y to be integral) we can achieve equality $\sum_{S\,:\,v\in S} \frac{1}{q} y'_S = x_v \ \ \forall v \in V$ where y' denotes the new decreased variables. This can be done as follows: if $\sum_{S\,:\,v\in S} \frac{1}{q} y'_S > x_v$ for some v, take an S containg x with $y_S > 0$ and decrease y_S while simultaneously increasing $y_{S\setminus\{v\}}$. Once equality is achieved for every v, increase the $y'_\emptyset$ variable in order to also achieve equality $\sum \frac{1}{q} y''_S = 1$ (where y'' are the new variables). But this means we have written x as a convex combination $x = \sum_S (\frac{1}{q} y''_S)\mathbf{1}(S)$ with $\sum \frac{1}{q} y''_S = 1$, so $x \in STAB(G)$.

(iii) $\Rightarrow$ (iv). It is sufficient to show $\omega(\bar{G}) = \chi(\bar{G})$ instead of showing it for every induced subgraph because since $QSTAB(G)$ is an integral polytope, the face $QSTAB(G[S])$ is also integral. Since $QSTAB$ is integral, $\omega(\bar{G}) = \alpha(G) = \max\{\sum x_v \,|\, x \in QSTAB(G)\} = \max\{\sum x_v \,|\, x(C) \leq 1 \ \forall$ clique C, $x \geq 0\}$. Consider the (nonempty) face of $QSTAB$ defined by $F = \{x \in \mathbb{R}^V : x \in QSTAB(G), \sum x_v = \alpha(G)\}$. There must be some constraint of $QSTAB$ that is tight on this face, i.e. there exists a clique C such that $F \subseteq \{x \in \mathbb{R}^V : x(C) = 1\}$. For every stable set S in G with $|S| = \alpha(G)$ (i.e. every maximum stable set) we have $\mathbf{1}(S) \in F$ and so $S \cap C \neq \emptyset$. Now define $G \setminus C = G[V \setminus C]$ and note that $\alpha(G \setminus C) = \alpha(G) - 1$ because every maximum stable set intersects C. Also, $QSTAB(G \setminus C)$ is integral because $G \setminus C$ is an induced subgraph and is therefore perfect, and we have already shown (i) $\Rightarrow$ (iii). Therefore we can recursively apply the above reasoning to $G \setminus C$. After doing this $\alpha(G)$ times we have removed $\alpha(G)$ cliques that cover V. This means $\chi(\bar{G}) \leq \alpha(G)$.

But since $\alpha(G) = \omega(\bar{G}) \leq \chi(\bar{G})$ is true in general, we must have $w(\bar{G}) = \chi(\bar{G})$.

(iv) $\Rightarrow$ (i) There is nothing to prove here because we have already shown (i) $\Rightarrow$ (iv) and so (iv) $\Rightarrow$ (i) follows by taking complements. $\square$

After being open for several decades, a structural characterization of perfect graphs was finally proved, and is stated in the following theorem. We won't give the very lengthy proof here.

Theorem 3 (Strong Perfect Graph Theorem: Chudnovsky, Robertson, Seymour, Thomas 2002) *G is perfect iff it has no odd hole and no odd antihole of size at least 5.*

Recall that a hole is an induced subgraph that is a cycle, and an antihole is the complement of a hole. A refinement of this theorem gives an algorithm to decide whether G is perfect.

2 Computing the Max Weight Stable Set in a Perfect Graph

One potential strategy for finding the maximum weight stable set in a perfect group is to maximize $w \cdot x$ over $QSTAB(G)$. However, $QSTAB$ has exponentially-many constraints (because there are exponentially-many cliques). This is not necessarily a problem because if we could solve the separation problem then we could use the ellipsoid method. But the separation problem iss hard in this case because if you were to try to find the constraint that is most violated by some $x \in \mathbb{R}^V$ you would need to look for the maximum weight clique with respect to the cost function x. This is equivalent to finding the maximum weight stable set in $\bar{G}$. Note that we have reduced the max weight stable set problem to the same problem in $\bar{G}$ and thus have not made progress. For general graphs it is in fact NP-hard to optimize over $QSTAB$.

Lovász found a solution in 1979, and this is the first use of semidefinite programming. He defined the Theta body $TH(G)$, a convex set in $\mathbb{R}^V$ that lies between $STAB$ and $QSTAB$ and can be efficiently optimized over. For any graph G (not necessarily perfect), $STAB(G) \subseteq TH(G) \subseteq QSTAB(G)$. Before we can define $TH(G)$ we need the following definition.

Definition 2 *An* orthonormal representation *of a graph G consists of a unit vector $c \in \mathbb{R}^N$ and, for each $i \in V$, a unit vector $u_i \in \mathbb{R}^N$ such that $v_i^T v_j = 0$ whenever $(i,j) \notin E$.*

Here the dimension N can take any value but there is no reason for N to exceed $|V|+1$ (because if $N > |V|+1$ then the $|V|+1$ vectors in the representation will lie in a proper subspace of $\mathbb{R}^N$). To motivate the definition of the Theta body, note that for any orthonormal representation, $\sum_{i \in V} (c^T u_i)^2 x_i \leq 1$ is a valid inequality for $STAB(G)$ because if x is the indicator vector $\mathbf{1}(S)$ for a stable set S then $\sum_{i \in V} (c^T u_i)^2 x_i = \sum_{i \in S} (c^T u_i)^2 \leq ||c||^2 = 1$ since $\{u_i : i \in S\}$ are orthonormal. The Theta body is defined as follows:

$$TH(G) = \left\{ x \in \mathbb{R}^V \ : \ \sum_{i \in V} (c^T u_i)^2 x_i \leq 1 \ \ \forall \text{ orthonormal representations } \{u_i\} \text{ of } G,\ x \geq 0 \right\}.$$

By its definition, we have $STAB(G) \subseteq TH(G)$. The Theta body is a convex set but is not necessarily polyhedral because it has infinitely many constraints, one for each orthonormal representation. In fact, we know how to characterize precisely when it is polyhedral.

Theorem 4 *The following are equivalent:*

(i) G is perfect,

(ii) $TH(G)$ is polyhedral,

(iii) $STAB(G) = TH(G)$,

(iv) $QSTAB(G) = TH(G)$.

We will not prove this theorem but we will finish the proof of the above claim that $STAB(G) \subseteq TH(G) \subseteq QSTAB(G)$ for all G. We already showed that $STAB(G) \subseteq TH(G)$ because the constraints defining $TH(G)$ are valid for indicator vectors of stable sets. Now we show that $TH(G) \subseteq QSTAB$. Let $x \in TH(G)$ and let C be any clique in G. We need to show $x(C) \leq 1$. Define an orthonormal representation by letting $u_i = c$ for $i \in C$ and letting the vectors $\{u_i : i \notin C\}$ be pairwise orthogonal and contained in $c^\perp$ (the subspace orthogonal to c). Now we have $x(C) = \sum_{i \in C} x_i = \sum_{i \in V} (c^T u_i)^2 x_i \leq 1$, completing the proof.

Next time we will talk about how to optimize over $TH(G)$ using semidefinite programming.

References

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