18.438 Advanced Combinatorial Optimization Michel X. Goemans

Problem Set 4

April 24th, 2012

This problem set is due in class on May 1st, 2012.

1. Consider the scribe notes at `http://math.mit.edu/~goemans/18438F09/lec16.pdf` on graph orientation using matroid intersection. Show that as stated the definition of $\mathcal{M}_2$ in the middle of page 16-2 does not give a matroid (rather, as mentioned in lecture, one has to specify the bases by imposing a cardinality constraint).

2. Consider a submodular function $f : 2^V \to \mathbb{R}$. Let $\mathcal{F} = \{S \subseteq V \mid |S| \equiv 1 \pmod 2\}$ be the family of odd sets and assume that $|V|$ is even. Let S^* be a minimal set minimizing f over $\mathcal{F}$. Show that there exist $a, b \in V$ such that S^* is the unique minimal set minimizing f over $\mathcal{C}_{ab} = \{S \subset V | a \in S, b \notin S\}$. Derive from this an algorithm for finding S^* with a polynomial number of oracle calls to f.

3. Can you find an algorithm for minimizing a submodular function over *even* sets which are non-empty and not the entire set? This is harder than the previous exercise.

4. Consider the separation problem for the matching polytope, i.e. given $x \in \mathbb{R}^E$, decide if $x \geq 0$, $x(\delta(v) \leq 1$ for all $v \in V$ and $x((E(S)) \leq (|S| - 1)/2$ for all odd sets S.

   (a) Show to use submodular function minimization to solve the separation problem efficiently.

   (b) Can you use a maximum flow problem to solve each submodular function minimization problem that arises?