18.438 Advanced Combinatorial Optimization Michel X. Goemans

Problem Set 3

April 5th, 2012

This problem set is due in class on April 12th, 2012.

1. Given a full rank matrix $A \in \mathbb{R}^{n\times n}$ (true also for any field F), let R and C denote the indices of the rows and columns of A. Given $I \subset R$, show using matroid intersection that there exists $J \subset C$ with $|I| = |J|$ such that both $A(I, J)$ and $A(R \setminus I, C \setminus J)$ are of full rank.

2. Suppose we are given an undirected graph $G = (V, E)$, and additional vertex $s \notin V$, an integer k, and we would like to add the minimum number of edges between s and vertices of V (multiple edges are allowed) such that the resulting graph H on $V + s$ has k edge-disjoint paths between any two vertices of V (i.e. the only cut that could possibly have fewer than k edges is the cut separating s from V).

   (a) Argue that this problem is equivalent to finding $x : V \to \mathbb{Z}_+$ minimizing $x(V)$ such that $\forall \emptyset \neq S \subset V$:
   $$x(S) \geq k - d_E(S),$$
   where $d_E(S) = |\delta_E(S)|$ corresponds to the number of edges between S and $V \setminus S$ in G.

   (b) Add k edges between s and each vertex of V. Let A be these $k|V|$ newly added edges. Say that $F \subseteq A$ is feasible if the graph $(V + s, E \cup (A \setminus F))$ has at least k edge-disjoint paths between any two vertices of V. Prove that the feasible sets form the independence sets of a matroid.

   (c) How would you efficiently solve the original problem (say that k is part of the input)?

3. Consider a matroid $M = (S, \mathcal{I})$, and let B be a given basis of M. Let $A = S \setminus B$. From M and B, we define a *linking system* $\mathcal{P}$ by
   $$\mathcal{P} = \{(A \cap B', B \setminus B') \subset A \times B : B' \text{ is a basis of } M\}.$$
   $\mathcal{P}$ consists of pairs $(X, Y) \subset A \times B$ which corresponds to valid basis exchanges for B in the matroid M. For such a pair (X, Y), we say that X is linked to Y. Observe that

   (a) $(\emptyset, \emptyset) \in \mathcal{P}$

   (b) $(X, Y) \in \mathcal{P} \Rightarrow |X| = |Y|$

   (We could add some additional axioms that would then characterize linking systems but we won't do it here.)

   (a) Given a bipartite graph $G = (V, E)$ with bipartition (A, B), let $\mathcal{P}$ be the pairs (X, Y) with $X \subseteq A$ and $Y \subseteq B$ such that there exists a perfect matching between X and Y. Show that $\mathcal{P}$ define a linking system.

   (b) Given a matrix L (over some field F), let A index the rows of L and B index the columns of L. Say that $X \subseteq A$ is linked to $Y \subseteq B$ is the corresponding submatrix $L(X, Y)$ is of full rank (over F). Show that this also define a linking system.

   (c) Let $\mathcal{P}_1 \subseteq 2^{A\times B}$ and $\mathcal{P}_2 \subseteq 2^{B\times C}$ be two linking systems. Define
   $$\mathcal{P}_1 * \mathcal{P}_2 = \{(X, Z) \subseteq A \times C : \exists Y \subseteq B \text{ with } (X, Y) \in \mathcal{P}_1 \text{ and } (Y, Z) \in \mathcal{P}_2\}.$$
   Show that $\mathcal{P}_1 * \mathcal{P}_2$ is also a linking system.

(d) Suppose we are given disjoint sets $V_0, V_1, \cdots, V_k$ and, for $i = 1, \cdots, k$, a linking system $\mathcal{P}_i$ on (V_{i-1}, V_i). This constitutes a linking network. Define a flow to be $(X_0, X_1, \cdots, X_k) \subseteq (V_0, V_1, \cdots, V_k)$ where X_{i-1} is linked to X_i in $\mathcal{P}_i$ for $i = 1, \cdots, k$. The value of the flow is $|X_0| = |X_1| = \cdots = |X_k|$. (If all the linking systems involved are of the matching type given above, a flow corresponds to a set of vertex-disjoint directed paths in a layered network. But the beauty here is that you can have many different types of linking systems involved.) How would you efficiently find a maximum flow (i.e. one of maximum value) in such a linking network (given access to matroid independence oracles for all the matroids defining the linking systems)? (Can you do it with matroid union/partition?)

(One can also derive a max-flow min-cut type result, but I won't formulate it here.)