18.438 Advanced Combinatorial Optimization Michel X. Goemans

Problem Set 2

March 13, 2012

This problem set is due in class on March 22, 2012. (There is no class on Tuesday March 20th.)

1. We have seen in lecture that any rational polyhedral cone C has an integral Hilbert basis. Assume that C is also pointed (i.e. there exists a vector $b \in \mathbb{R}^n$ such that $b^T x > 0$ for all $x \in C \setminus \{0\}$). Show then that

$$H := \{a \in (C \setminus \{0\}) \cap \mathbb{Z}^n | a \text{ is not the sum of two other integral vectors in } C\}$$

   is the *unique minimal* Hilbert basis of C (i.e. it is a Hilbert basis, and every other Hilbert basis contains all vectors in H).

2. Prove that a $0, \pm 1$ matrix which is minimally *not* totally unimodular (i.e. having all square subdeterminants in $\{-1, 0, 1\}$ except for the matrix itself) has determinant ± 2.

   (Hint: think about pivoting on a nonzero element.)

3. Prove that if a matroid is representable over GF(2) and over GF(3) then it can be represented over any field by a totally unimodular matrix (it is thus regular).

   (Hint: Start with a representation $[I|B]$ of the matroid over GF(3) and interpret it as a real matrix with entries $0, \pm 1$. Also use the previous exercise.)

   (Remark: The statement is still true if one replaces GF(3) by any field of charateristic other than 2.)

4. Given a graph $G = (V, E)$, let $\mathcal{I} = \{F \subseteq E | \, |E(S) \cap F| \leq 2|S| - 3 \text{ for all } S \subseteq V \text{ with } |S| > 1\}$. We want to show that $(E, \mathcal{I})$ defines a matroid.

   (a) For $F \in \mathcal{I}$, let

$$f_F(S) := \begin{cases} 2|S| - 3 - |E(S) \cap F| & |S| > 1 \\ 0 & |S| \leq 1 \end{cases}$$

   Observe that $F \in \mathcal{I}$ is equivalent to f_F being nonnegative. Observe that the function f_F is *not* submodular.

   For $F \in \mathcal{I}$, call a set $S \neq \emptyset$ tight if $f_F(S) = 0$. Show that if A and B are tight and $|A \cap B| > 1$ then $A \cup B$ is tight (and so is $A \cap B$ but you won't need that).

   (b) Prove that $(E, \mathcal{I})$ satisfy the matroid axioms (for independence).