Problem 1 (Based on Zeyuan Zhu’s solution)

We first claim $H$ is a Hilbert basis. Assume that there is an $x \in \mathbb{C}\{0\} \cap \mathbb{Z}^n$ which cannot be represented in $H$. Among these vectors, let $x$ minimize $b^T x$. Which can be done because $b^T y > 0$ for all $y \in \mathbb{C}\{0\}$ and $\mathbb{Z}^n$ is discrete. Now by definition $x$ is not in $H$, hence $x = y + z$ can be written as a sum of two other elements. However since $b^T x = b^T y + b^T z > b^T y$ and $b^T x > b^T z$ for the same reason. We know that both $y$ and $z$ can be represented in $H$ because of the minimality of $b^T x$. So we have a contradiction.

The fact that $H$ is minimal is straightforward. Let $x \in H$, write $x$ as a combination of some elements in $H'$. The definition of $H$ implies that there are at most one such element. so $x \in H'$.

Problem 2

Solution 1

Let $A$ be such a matrix. We prove by induction on the size $n$.

For $n=2$, the result is trivial. For induction step, we pivot on the corner element $a_{11}$, without loss of generality assuming $a_{11} = 1$. We do row operations on $A$. That is keeping the first row, adding to the rest to make $a_{k1} = 0$ for $k = 2, 3, ..., n$. Then we get a new matrix

$$A' = \begin{vmatrix} 1 & \ast \\ 0 & M' \end{vmatrix}$$

Claim: $M$ is still a minimally not totally unimodular matrix.

For any proper square submatrix $N'$ of $M'$, we add the corresponding elements in the first row and the first column to $N'$ to get $N$, then

$$\det(N') = \det(N)$$

Just note that if we reverse the row operations on $N$, it goes back to an square submatrix of $A$ hence the determinant of the submatrix is in $\{-1, 0, 1\}$. Therefore $\det(N')$ is also in $\{-1, 0, 1\}$. Furthermore note that $\det(M')=\det(A)\neq \pm 1, 0$, so we prove the claim.

By induction hypothesis, $\det(M')= \pm 2$, so is $\det(A)$.

Solution 2 (Based on Zeyuan Zhu’s solution)

We again prove by induction. The case of $n=2$ is trivial. Without loss of generality we assume $a_{11} \neq 0$, and for simplicity we assume that $a_{11} = 1$. Using "Chio Pivotal Condensation method", the determinant of $A$ is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{11} & a_{14} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{21} & a_{24} \\ a_{11} & a_{12} & a_{13} & \cdots & a_{11} & a_{1n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{31} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{11} & a_{12} & a_{13} & \cdots & a_{11} & a_{1n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n1} & a_{nn} \end{vmatrix}$$

Now, we look at the new $(n-1) \times (n-1)$ matrix above. It is also minimally not tally unimodular:

- its determinant is not within $\{-1, 0, 1\}$ because $\det(A)$ is not
• any \( m \times m \) submatrix of it has the same determinant as a \( (m + 1) \times (m + 1) \) submatrix of the original matrix \( A \), and thus must be in \( \{-1, 0, 1\} \).

Therefore by induction, \( \det(A) \) is either 2 or -2.

**Problem 3 (Based on Zeyuan Zhu’s solution)**

Suppose that for our given matroid \( M \) we have a representation \([I|M]\) over \( GF(3)\), which means \( M \) contains entries of \( \{\pm 1, 0\} \) only. Now I first claim that, if we define \( M' \) to be such that it is 1 whenever \( M \) is \( \pm 1 \) and 0 if \( M \) is 0, then:

**Lemma 1.** \([I|M']\) is the representation of the same matroid \( M \) over \( GF(2)\).

**Proof.** This proof is similar to the proof we have seen in class that says any binary matroid does not have a \( U_2^2 \) minor.

Indeed, let \( N \) be the matroid defined by \([I|M']\) over \( GF(2)\). Then, \( M \) and \( N \) share a same basis \( B \) on coordinates corresponding to the columns in the diagonal matrix \( I \).

This set \( B \) is a basis both in \( M \) and \( N \). Using the same technique we have seen in class, that is, let \( X \) be the set that minimizes \(|X\Delta B|\). If \(|X\Delta B| > 2\), then this shows that \( N \) has a \( U_2^2 \) minor, but this cannot happen to a binary matroid \( N \). This means, we must have \(|X\Delta B| = 2\) if \( M \neq N \).

But this is also impossible, because as seen in class, for each \( b \in B \) and \( s \in S \setminus B \), \( B - b + s \) is an independent in \( M \) iff it is independent in \( N \). This means, we must have \( M = N \), and thus \([I|M']\) is the binary representation of the same matroid.

Now back to the original proof. This time, let \( M \) be the given matroid (representable as \([I|M]\) over \( GF(3)\) or \([I|M']\) over \( GF(2)\)), but let us use \( P \) to denote the matroid representable by \([I|M]\) over reals. I want to show that:

**Lemma 2.** \( M = P \) are the same matroid.

**Proof.** For any subset \( T \subseteq S \), if \( T \) is independent in \( M \), let \( M_T \) be the submatrix of \([I|M]\) of size \(|T| \times |T|\) that has determinant not equal to 0 \( \in F_3 \), and thus also not equal to 0 over reals, so \( T \) is independent in \( P \).

We can actually say something stronger here, that is, \( \det(M_T) \) is not only nonzero over reals, but also \( \pm 1 \). This can be shown by induction. If \(|T| = 1\) this is true for sure because all entries of \( M \) are \( \pm 1 \). For any \( T \), by induction we know that all its subdeterminants are \( \pm 1 \), so if \( \det(T) \not\in \{-1, 1\} \), then \( \det(T) \) must be \( \pm 2 \) by Problem 2. However, since the same matrix \( M_T \) (after converting to \( M_T' \) by changing \(-1\) to \( 1 \)) is also full rank over \( GF(2) \), we must have \( \det(T) \neq \pm 2 \). This says we can only have \( \det(T) = \pm 1 \). This property will be used in the next paragraph.

If \( T \) is a circuit (smallest dependent set) in \( M \), we similarly define \( M_T \) to be some \(|T| \times |T|\) submatrix of \([I|M]\), such that its determinant over \( GF(3) \) is 0 (because \( T \) is dependent), but its subdeterminants over all smaller matrices are \( \pm 1 \in F_3 \) (because \( T \) is a circuit). Now, we know that all subdeterminants of \( M_T \) over real are also \( \pm 1 \) (this is because of the previous paragraph). Great! Now we can use Problem 2 again and claim that \( \det(M_T) \in \{\pm 2, \pm 1, 0\} \) over reals, but since it must be 0 module 3, we must have \( \det(M_T) = 0 \) over reals.

In sum, we have shown that all independent sets are also independent in \( P \); and also all circuits are also circuits in \( P \). This ends the proof that \( M = P \).

Back to the original problem, since we now know that the given matroid is representable over real by a unimodular matrix, and further more, if one carefully follows the proof above, he should notice that any independent set corresponds to determinant \( \pm 1 \) and circuit corresponds to determinant 0. This must also be true for any field with a unit element.
Problem 4 (Based on Mohammad Bavarian’s solution)

4(a)
Observe that $|A \cap B| > 1$ implies $|A| > 1, |B| > 1$ and $|A \cup B| > 1$. We have

$$f_F(A \cap B) + f_F(A \cup B) = 2|A \cap B| - 3 - |F \cap E(A \cap B)| + 2|A \cup B| - 3 - |F \cap E(A \cup B)|$$

$$\leq 2|A| + 2|B| - 6 - |F \cap E(A)| - |F \cup E(B)| = 0$$

Hence $A \cup B$ is tight.

4(b)
Consider $|F_1| < |F_2|$. We would like to show that there is an $e \in F_2 \setminus F_1$ such that $F_1 \cup \{e\}$ is still independent.

For each $e = (u, v) \in F_2$ define $V_e \subseteq V$ such that $|V_e|$ is maximal with property that $e \in V_e$ and $f_{F_1}(V_e) = 0$. We are basically taking maximal tight set w.r.t $F_1$ containing $e$.

Now for $e = (u, v) \in F_2 \cap F_1$ clearly at least one tight set containing $e$ exists. That is $S = \{u, v\}$ and $f_{F_1}(S) = f_{F_2}(S) = 2 \times 2 - 3 - 1 = 0$.

If for some $e = (u, v) \in F_2 \setminus F_1$, no tight set containing $e$ exists, we are done. Because one can verify that $f_{\{e\} \cup F_1}(S) \geq 0$ for any $S \subseteq V$.

Next we assume for any $e \in E$, the set $V_e$ exists and is nonempty. Take all the different $V_{e_i}, i = 1, 2, ..., k$. Then $V_{e_i}$ induce a partition of the edges of $F_1$.

Indeed assume that $(E(V_{e_i}) \cap F_1) \cap (E(V_{e_j}) \cap F_1) \neq \emptyset$ for some $i \neq j$. Then $|V_{e_i} \cap V_{e_j}| \geq 2$ because they both contain the vertices of an edge in $F_1$. According to part (a), $V' = V_{e_i} \cup V_{e_j}$ is also tight w.r.t $F_1$, this is impossible by the maximality of $V_{e_i}$ and $V_{e_j}$.

Now note that

$$f_{F_1}(V_{e_i}) = 0 \leq f_{F_2}(V_{e_i})$$

one has

$$|E(V_{e_i}) \cap F_1| \geq |E(V_{e_i}) \cap F_2|$$

and

$$|F_1| = \sum |E(V_{e_i}) \cap F_1| \geq \sum |E(V_{e_i}) \cap F_2| \geq |F_2|$$

contradiction!