18.438 Advanced Combinatorial Optimization Michel X. Goemans

Problem Set 1

February 14, 2012

This problem set is due in class on February 28, 2012.

1. In class we argued that Edmonds' (cardinality matching) algorithm runs in $O(mn^2)$ time (where $m = |E|$ and $n = |V|$). You are asked to improve this to $O(n^3)$ by arguing that the time taken between two augmentations can be reduced to $O(n^2)$. (Hint: One should maintain the labels Even, Odd, and Free we discussed in the proof of Edmonds-Gallai and not start from scratch when detecting a blossom.)

2. Show that a 2-connected factor-critical graph $G = (V, E)$ admits a *proper odd* ear decomposition starting from an odd cycle. Deduce from this that any 2-connected factor critical graph has $|E|$ *linearly independent* near perfect matchings (i.e. of size $(|V| - 1)/2$). (By linearly independent, it is meant that their characteristic vectors are linearly independent.)

3. Given a graph $G = (V, E)$ and an even subset $T \subseteq V$, a T-join is a set $F \subseteq E$ such that, for every vertex $v \in V$, the degree $d_F(v)$ of v in F is odd if and only if $v \in T$. Let $\tau(G; T)$ denote the minimum cardinality T-join in G. Assume G is 2-edge-connected. Among all ear-decompositions of G, consider one for which the number of *even* ears is minimum (recall that an ear is even if it contains an even number of edges); let $\epsilon(G)$ be this minimum number of even ears.

   Prove that for all even $T \subseteq V$: $\tau(G; T) \leq \frac{1}{2}(|V| - 1 + \epsilon(G))$.

   [There is actually a min-max relationship here:

   $$\max_{\text{even } T} \tau(G, T) = \frac{1}{2}(|V| - 1 + \epsilon(G))$$

   but I don't know a short proof of it. If you can think of a short proof of it, I'd be happy to know about it.]

4. Prove that a cubic graph has a nowhere zero 3-flow if and only if the graph is bipartite.