18.438 Advanced Combinatorial Optimization	September 29, 2009
Lecture 6	
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In this lecture, we will focus on Total Dual Integrality (TDI) and its application to the matching polytope. We will also introduce the notion of a Hilbert basis and point out its connection to TDI.

1 The Matching Polytope

Given an undirected graph $G = (V, E)$, a *matching* $M \subseteq E$ is a subset of edges such that no two edges in M share a common vertex. We can identify M with its incidence vector:

$$\chi(M) \in \mathbb{R}^{|E|} \quad : \quad (\chi(M))_e = \begin{cases} 1 & \text{if } e \in M, \\ 0 & \text{otherwise.} \end{cases}$$

We define the *matching polytope* of G, $\mathcal{P} = \mathcal{P}(G)$ to be the convex hull of these incidence vectors, i.e.

$$\mathcal{P}(G) = conv\{\chi(M) : M \text{ is a matching of } G\}.$$

Note that since the number of matchings in G is finite, $\mathcal{P}(G)$ is a convex polytope.

Our goal is to represent $\mathcal{P}$ by a set of linear inequalities defined on a set of $|E|$ variables, $\{x_e \in \mathbb{R}\}_{e \in E}$. We must have $x_e \geq 0$, $\forall e \in E$. Also, every vertex can have at most one adjacent edge in any matching, i.e.

$$x(\delta(v)) \triangleq \sum_{e \in \delta(v)} x_e \leq 1,$$

where $\delta(v)$ is the set of edges incident on vertex v. Thus our first attempt at a linear description of $\mathcal{P}$ is

$$P_1 = \left\{ (x_e \in \mathbb{R})_{e \in E} \ : \ \begin{array}{ll} x_e \geq 0 & \forall e \in E \\ x(\delta(v)) \leq 1 & \forall v \in V \end{array} \right\}.$$

Since P_1 is a convex subset of $\mathbb{R}^{|E|}$ and $\chi(M) \in P_1$ for each matching M, it follows from the definition of convex hull that $\mathcal{P} \subseteq P_1$. However, as illustrated by the following example, $\mathcal{P} \subsetneq P_1$ in general since P_1 can have non-integral extreme points. Consider the triangle (K_3)—its matching polytope is

$$\mathcal{P} = conv\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}.$$

The point $(0.5, 0.5, 0.5) \in P_1$, i.e. it satisfies the constraints above; however it is not in the convex hull of the matching vectors.

The above example motivates the following family of additional constraints (introduced by Edmonds). Observe that for any matching M, the subgraph induced by M on any odd cardinality vertex subset U has at most $(|U| - 1)/2$ edges. Thus, without losing any of the matchings, we can introduce the following additional constraints:

$$x(E(U)) \triangleq \sum_{e \in E(U)} x_e \leq \frac{|U| - 1}{2}, \qquad U \subseteq V, \ |U| \text{ is odd},$$

where $E(U)$ is the set of edges in the subgraph induced by G on U. These constraints are called the *odd set constraints* or *blossom constraints*. For the triangle, taking $U = V = \{1, 2, 3\}$, we get the constraint $x_1 + x_2 + x_3 \leq 1$. This constraint is violated by the point $(0.5, 0.5, 0.5)$. Thus, our second attempt at a linear description of the matching polytope is

$$P_2 = \left\{ (x_e \in \mathbb{R})_{e \in E} \;:\; \begin{array}{ll} x_e \geq 0 & \forall e \in E \\ x(\delta(v)) \leq 1 & \forall v \in V \\ x(E(U)) \leq \frac{|U|-1}{2} & \forall U \subseteq V \;:\; |U| \text{ is odd} \end{array} \right\}.$$

The following theorem asserts that this description indeed captures the matching polytope.

Theorem 1 (Edmonds, 1965) *P_2 is identical to the Matching polytope, i.e. $\mathcal{P} = P_2$.*

Edmonds gave an algorithmic proof for this theorem; instead, we will prove it over the course of this and the next lecture using the concept of *Total Dual Integrality* (TDI).

2 Total Dual Integrality

Recall the standard formulations of a primal and its dual linear program.

$$(\text{Primal } (P)) \quad \left\{ \begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax \leq b \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{ll} \min & b^\top y \\ \text{s.t.} & A^\top y = c \\ & y \geq 0 \end{array} \right\} \quad (\text{Dual } (D))$$

We define Total Dual Integrality as follows.

Definition 1 (Total Dual Integrality) *A linear system $\{Ax \leq b\}$ (with A and b rational) is Totally Dual Integral (TDI) if for any integral (cost) vector $c \in \mathbb{Z}^n$ for the primal, such that $\max(c^\top x, Ax \leq b)$ is finite (i.e. the primal has a solution), there exists an optimal dual solution $y \in \mathbb{Z}^m$.*

To establish the connection between TDI and Theorem 1, we state the following theorem (we give a proof later).

Theorem 2 (Edmonds-Giles, 1979) *If a linear system $\{Ax \leq b\}$ is TDI, and b is integral, then $\{Ax \leq b\}$ is integral, i.e. all its extreme points are integral.*

This theorem implies that if we can prove that the linear system P_2 is TDI (we will prove this in the next lecture), then all the extreme points of P_2 are integral. For rational linear systems, this is equivalent to the polyhedron P_2 being the convex hull of all integral points contained in it. Hence, this will prove Theorem 1.

It is important to note that TDI is not a property of the polyhedron, but of its representation. In fact, the following theorem states that any rational polyhedron has a TDI representation.

Theorem 3 (Edmonds-Giles, 1979) *Let P be a rational polyhedron. Then, $\exists A, b$ such that $P = \{x \;:\; Ax \leq b\}$, $\{Ax \leq b\}$ is TDI and A is integral.*

To illustrate this point, consider the two-dimensional polytope (refer to Figure 1) defined as

$$\mathcal{P} = conv\{(0,3), (2,2), (0,0), (3,0)\}.$$

This polytope may have many different representations. For example,

$$\mathcal{P} = \left\{ \begin{array}{l} x \geq 0, \;\; y \geq 0 \\ x + 2y \leq 6 \\ 2x + y \leq 6 \end{array} \right\}.$$

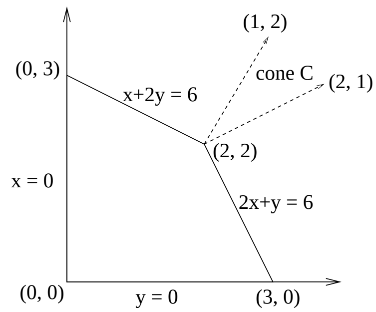

Figure 1: A primal linear system and a dual cone.

This linear system, however, is not TDI. For example, if the cost vector is $c = (1\ 1)^\top$, then the primal maximum is achieved by $(2, 2)$. However, $(1, 1)$ cannot be expressed as a linear integer combination of $(1, 2)$ and $(2, 1)$, the normals to the tight constraints at $(2, 2)$. Thus, there is no integral dual optimum and $\mathcal{P}$ is not TDI.

In Theorem 3, we should emnphasize that A is integral, but of course b will only be integral if P itself is integral, see Theorem 2. In the rest of the lecture, we will prove Theorems 2 and 3.

3 Hilbert Basis

We now need to introduce the concept of a *Hilbert basis.*

Definition 2 *A set of vectors* $\{a_1, a_2, \ldots, a_k\}$, $a_i \in \mathbb{Z}^n\ \forall i$, *defines a* Hilbert basis *if for any* $x \in C \cap \mathbb{Z}^n$, *where*

$$C = cone(a_1, a_2, \ldots, a_k) = \left\{ \sum_i \lambda_i a_i : \lambda_i \geq 0,\ \ \lambda_i \in \mathbb{R}\ \forall i \right\},$$

there exists $\mu_1, \mu_2, \ldots, \mu_n$, *such that* $\mu_i \in \mathbb{Z}$ *and* $\mu_i \geq 0$ *for each* i, *and* $x = \sum_i \mu_i a_i$.

The following theorem, then, is a simple consequence of LP duality.

Theorem 4 *A linear system* $\{Ax \leq b\}$ *is TDI iff for each face* F *of* $P = \{x : Ax \leq b\}$, *the normals to the tight constraints for* F *form a Hilbert basis.*

In the above theorem, we could have replaced 'each face' by 'each extreme point', and the proof would also follow easily from LP duality, since for every vector c, there always exists an optimum extreme point.

In our previous example (refer to Figure 1), a Hilbert basis for the cone (the *dual cone* associated with the vertex $(2, 2)$) defined by the vectors $(1, 2)$ and $(2, 1)$ is given by the set of vectors $H = \{(1, 2), (2, 1), (1, 1)\}$. We can get the additional vector $(1, 1)$ by adding the redundant constraint $x_1 + x_2 \leq 4$ in the primal.

In fact, by considering also the dual cones corresponding to the vertices $(3, 0)$, $(0, 3)$ and $(0, 0)$, one can show that the linear system

$$\left\{ \begin{array}{lcl} x_1, x_2 & \geq & 0 \\ x_1 + 2x_2 & \leq & 6 \\ 2x_1 + x_2 & \leq & 6 \\ x_1 + x_2 & \leq & 4 \\ x_1 & \leq & 3 \\ x_2 & \leq & 3 \end{array} \right.$$

is TDI. For example, the cone corresponding to the vertex $(3, 0)$ has a Hilbert basis $\{(1, 2), (-1, 0), (0, 1)\}$.

The following theorem, in combination with Theorem 4, proves Theorem 3.

Theorem 5 *Any rational polyhedral*[1] *cone C has a finite integral Hilbert basis.*

Proof: Let $C = \{\sum_i \lambda_i a_i \ : \ \lambda_i \geq 0, \ \lambda_i \in \mathbb{R}\}$, $a_i \in \mathbb{Z}^n$. Define $Q = \{\sum_i \lambda_i a_i \ : \ 0 \leq \lambda_i \leq 1\}$. For any $c \in C \cap \mathbb{Z}^n$,

$$c = \sum_i \lambda_i a_i = \sum_i (\lambda_i - \lfloor \lambda_i \rfloor) a_i + \sum_i \lfloor \lambda_i \rfloor a_i = z + w,$$

where $z = \sum_i (\lambda_i - \lfloor \lambda_i \rfloor) a_i$ and $w = \sum_i \lfloor \lambda_i \rfloor a_i$. Since $a_i \in \mathbb{Z}^n$ and $\lfloor \lambda_i \rfloor \in \mathbb{Z}$ for each i, $w \in \mathbb{Z}^n$. Since $c \in \mathbb{Z}^n$, this implies that $z \in \mathbb{Z}^n$. Clearly, $z \in Q$; hence, $z \in Q \cap \mathbb{Z}^n$. Furthermore, each $a_i \in Q \cap \mathbb{Z}^n$. Hence, c is an integral combination of vectors in $Q \cap \mathbb{Z}^n$. Thus, $Q \cap \mathbb{Z}^n$ is a Hilbert basis for C. $\square$

We now give a proof of Theorem 2.

Proof of Theorem 2: We proceed by contradiction. Consider an extreme point x^* of P such that $x^*_j \notin \mathbb{Z}$ for some j. We can find an integral vector c such that x^* is the unique optimal solution corresponding to c by picking a rational vector c in the interior of the dual cone (always full-dimensional) of x^* and scaling appropriately. Consider $\hat{c} = c + \frac{1}{q} e_j$ where q is an integer. Since the cone is full dimensional, $\hat{c}$ will be in the interior of the dual cone of x^* for a sufficiently large q. Now it follows that $(q\hat{c})^\top x^* - (qc)^\top x^* = x^*_j \notin \mathbb{Z}$. This means that at least one of $(q\hat{c})^\top x^*$ and $(qc)^\top x^*$ is not integral. By duality and the fact that b is integral, we conclude that one of the two corresponding dual optimal solutions (say y and $\hat{y}$) is not integral. This contradicts the TDI property since both qc and $q\hat{c}$ are integral. $\square$

[1] i.e. generated by a finite number of vectors