18.438 Advanced Combinatorial Optimization	Michel X. Goemans

Problems

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This list of exercises will be updated throughout the term. You need to solve a fraction (to be updated, say 50% for now) of them by December 3rd, 2009.

1. (Corrected.) Show that a 2-connected factor-critical graph $G = (V, E)$ admits a *proper odd* ear decomposition starting from an odd cycle. Deduce from this that any 2-connected factor critical graph has $|E|$ *linearly independent* near perfect matchings (i.e. of size $(|V|-1)/2$). (By linearly independent, it is meant that their characteristic vectors are linearly independent.)

2. (Corrected.) Consider the matching polytope $P(G)$ of a graph $G = (V, E)$. Show that a blossom constraint
$$\sum_{e \in E(S)} x_e \leq \frac{|S| - 1}{2}$$
induces a facet of $P(G)$ different from those induced either by $x_e \geq 0$ for some $e \in E$ or by $x(\delta(v)) \leq 1$ for some $v \in V$ if and only if the graph $(S, E(S))$ is 2-connected and factor-critical.

   (If you are not familiar with polyhedral arguments, ask me.)

3. Show that every (undirected) graph $G = (V, E)$ with a Hamiltonian cycle (a cycle that contains all vertices) has a nowhere zero 4-flow.

4. Show that a graph $G = (V, E)$ has a nowhere zero k-flow if and only if there exists an orientation $D = (V, A)$ of G such that, for every $S \subset V$, $|\delta^+(S)| \geq \frac{1}{k}|\delta(S)|$, where $\delta(S)$ denotes the edges of G between S and $V \setminus S$ and $\delta^+(S)$ denotes the arcs of D from S to $V \setminus S$.

5. We have seen in lecture that any rational polyhedral cone C has an integral Hilbert basis. Assume that C is also pointed (i.e. there exists a vector $b \in \mathbb{R}^n$ such that $b^T x > 0$ for all $x \in C \setminus \{0\}$). Show then that
$$H := \{a \in (C \setminus \{0\}) \cap \mathbb{Z}^n | a \text{ is not the sum of two other integral vectors in } C\}$$
is the *unique minimal* Hilbert basis of C (i.e. it is a Hilbert basis, and every other Hilbert basis contains all vectors in H).

6. Given a graph $G = (V, E)$, let $\mathcal{I} = \{S \subseteq V :$ there exists a matching M covering S (and possibly other vertices)$\}$. Show that $(V, \mathcal{I})$ defines a matroid.

7. Given a graph $G = (V, E)$, let $\mathcal{I} = \{F \subseteq E | \, |E(S) \cap F| \leq 2|S| - 3 \text{ for all } S \subseteq V \text{ with } |S| > 1\}$. Show that $(E, \mathcal{I})$ defines a matroid.

8. Given a full rank matrix $A \in \mathbb{R}^{n \times n}$ (true also for any field F), let R and C denote the indices of the rows and columns of A. Given $I \subset R$, show using matroid intersection that there exists $J \subset C$ with $|I| = |J|$ such that both $A(I, J)$ and $A(R \setminus I, C \setminus J)$ are of full rank.

9. Consider a submodular function $f : 2^V \to \mathbb{R}$. Let $\mathcal{F} = \{S \subseteq V | \, |S| \equiv 1 \pmod 2\}$ and assume that $|V|$ is even. Let S^* be a minimal set minimizing f over $\mathcal{F}$. Show that there exist $a, b \in V$ such that S^* is the unique minimal set minimizing f over $\mathcal{C}_{ab} = \{S \subset V | a \in S, b \notin S\}$. Derive from this an algorithm for finding S^* with a polynomial number of oracle calls to f.

10. Can you find an algorithm for minimizing a submodular function over *even* sets which are non-empty and not the entire set? This is harder than the previous exercise.

11. (corrected on 11/30/09) Suppose we are given an undirected graph $G = (V, E)$, and additional vertex $s \notin V$, an integer k, and we would like to add the minimum number of edges between s and vertices of V (multiple edges are allowed) such that the resulting graph H on $V + s$ has k edge-disjoint paths between any two vertices of V (i.e. the only cut that could possibly have fewer than k edges is the cut separating s from V). Argue that this problem is equivalent to finding $x : V \to \mathbb{Z}_+$ minimizing $x(V)$ such that $\forall \emptyset \neq S \subset V$:

$$x(S) \geq k - d_E(S),$$

where $d_E(S) = |\delta_E(S)|$. Show that an optimum x can be found in polynomial time (either use a heavy hammer, or do it delicately with one finger...).

(Hint: Think submodularity... One possibility is to introduce k edges between s and every vertex of V, and consider all the possible subsets *to delete* that would make the graph appropriately edge connected. Any nice structure?)

12. Using the previous problem with $k \geq 2$, derive an efficient algorithm for finding a minimum number of edges to add to $G = (V, E)$ (without adding an additional vertex) so as to make the resulting multigraph k-edge-connected (there are no restrictions on which edges can be added).

(Argue that if $\gamma = \min x(V)$ in the previous problem then the minimum number of edges to add in this problem is precisely $\lceil \gamma/2 \rceil$.)

13. Given a graph $G = (V, E)$, pairs $\{s_1, t_1\}$ and $\{s_2, t_2\}$ of vertices, and capacities $c : E \to \mathbb{R}_+$. We would like to find d_1 and d_2 maximizing $d_1 + d_2$ and such that there is a multiflow of value d_1 between s_1 and t_2 and of value d_2 between s_2 and t_2. Prove that the max of $d_1 + d_2$ equals the minimum capacity of a cut separating both pairs $\{s_1, t_1\}$ and $\{s_2, t_2\}$ (thus either s_1, s_2 is on one side (and t_1, t_2 on the other), or s_1, t_2 is one of the side). Furthermore argue that the maximum multiflow can be chosen to be half integral.