1 Nonbipartite Matching

Our first topic of study is matchings in graphs which are not necessarily bipartite. We begin with some relevant terminology and definitions. A matching is a set of edges that share no endvertices. A vertex \( v \) is covered by a matching if \( v \) is incident with an edge in the matching. A matching that covers every vertex is known as a perfect matching or a 1-factor (i.e., a spanning regular subgraph in which every vertex has degree 1). We will let \( \nu(G) \) denote the cardinality of a maximum matching in graph \( G \). A vertex cover is a set \( C \) of vertices such that every edge is incident with at least one vertex in \( C \). The minimum cardinality of a vertex cover is denoted \( \tau(G) \). The following simple proposition relates matchings and vertex covers.

**Proposition 1** If \( M \) is a matching and \( C \) is a vertex cover then \( |M| \leq |C| \).

**Proof:** For each edge in \( M \), at least one of the endvertices must be in \( C \), since \( C \) covers every edge. Since the edges in \( M \) do not share any endvertices, we must have \( |M| \leq |C| \). \( \Box \)

This proposition implies that \( \nu(G) = \max_M |M| \leq \min_C |C| = \tau(G) \), so \( \nu(G) \leq \tau(G) \). König showed that in fact equality holds if \( G \) is a bipartite graph with no isolated vertices. Unfortunately if \( G \) is not bipartite then we may have \( \nu(G) < \tau(G) \). For example, if \( G \) is the cycle on three vertices then \( \nu(G) = 1 \) but \( \tau(G) = 2 \). We will give another upper-bound for \( \nu(G) \) after introducing some more definitions.

If \( G = (V, E) \) is a graph and \( U \subseteq V \), \( G - U \) denotes the subgraph of \( G \) obtained by deleting the vertices of \( U \) and all edges incident with them. Let \( o(G - U) \) denote the number of components of \( G - U \) that contain an odd number of vertices. Let \( M \) be a matching in \( G - U \) and consider a component of \( G - U \) with an odd number of vertices. There must be at least one unmatched vertex \( v \) in this component, since any matching necessarily covers an even number of vertices. Treating \( M \) as a matching in \( G \), it is possible that we could increase the size of \( M \) by matching \( v \) with some vertex in \( U \). However, we can add at most \( |U| \) edges to \( M \) in this manner, since the vertices in \( U \) will eventually all be matched. Thus any matching in \( G \) must have at least \( o(G - U) - |U| \) unmatched vertices. This argument shows that the maximum size of a matching is upper-bounded by \( ([|V|] + |U| - o(G - U))/2 \), for any subset \( U \). The following theorem strengthens this result.

**Theorem 2 (Tutte-Berge Formula)** Let \( G = (V, E) \) be a graph. Then

\[
\nu(G) = \max_M |M| = \min_{U \subseteq V} ([|V| + |U| - o(G - U))/2,
\]

where the maximization is over all matchings \( M \) in \( G \).

**Proof:** We will consider the case that \( G \) is connected. If \( G \) is not connected, the result follows by adding the formulas for the individual components. The proof proceeds by induction on the order of \( G \). If \( G \) has at most one vertex then the result holds trivially. Otherwise, suppose that \( G \) has at least two vertices. We consider two cases.

**Case 1:** \( G \) contains a vertex \( v \) that is covered by all maximum matchings. The subgraph \( G - v \) cannot have a matching of size \( \nu(G) \), otherwise that would give a maximum matching for \( G \) that leaves \( v \) unmatched. Thus \( \nu(G - v) = \nu(G) - 1 \). By induction the result holds for the graph
$G - v$, so there exists a set $U' \subset V - v$ that achieves equality in the Tutte-Berge Formula. Defining $U = U' \cup \{v\}$, we see that

$$\nu(G) = \nu(G - v) + 1 = (|V - v| + |U'| - o(G - v - U'))/2 + 1 = ((|V| - 1) + (|U| - 1) - o(G - U))/2 + 1 = (|V| + |U| - o(G - U))/2$$

Case 2: For every vertex $v \in G$, there is a maximum matching that does not cover $v$. We will prove that each maximum matching leaves exactly one vertex uncovered. Suppose to the contrary, that is, each maximum matching leaves at least two vertices uncovered. We choose a maximum matching $M$ and its two uncovered vertices $u$ and $v$ such that we minimize $d(u, v)$, the distance between vertices $u$ and $v$. If $d(u, v) = 1$ then the edge $uv$ can be added to $M$ to obtain a larger matching, which is a contradiction.

Otherwise, $d(u, v) \geq 2$ so we may fix an intermediate vertex $t$ on some shortest $u$-$v$ path. By the assumption of the present case, there is a maximum matching $N$ that does not cover $t$. Furthermore, we may choose $N$ such that its symmetric difference with $M$ is minimal. If $N$ does not cover $u$ then $(N, u, t)$ contradicts our choice of $(M, u, v)$. Thus $N$ covers $u$ and, by symmetry, $v$ as well. Since $N$ and $M$ both leave at least two vertices uncovered, there exists a second vertex $x \neq t$ that is covered by $M$ but not by $N$. Let $xy$ be the edge in $M$ that is incident with $x$. If $y$ is also uncovered by $N$ then $N + xy$ is a larger matching than $N$, a contradiction. So let $yz$ be the edge in $N$ that is incident with $y$, and note that $z \neq x$. Then $N + xy - yz$ is a maximum matching that does not cover $t$ and has smaller symmetric difference with $M$ than $N$ does. This contradicts our choice of $N$, so each maximum matching must leave exactly one vertex uncovered. Then $\nu(G) = (|V| - 1)/2$.

The Tutte-Berge Formula then follows by choosing $U = \emptyset$.

A natural question to ask next is: Given a graph $G$, what is a set $U \subset V(G)$ giving equality in the Tutte-Berge Formula? Such a set is provided by the Edmonds-Gallai Decomposition of $G$.

This decomposition partitions $V(G)$ into three sets: $D(G)$ is the set of all vertices $v$ such that there is some maximum matching that leaves $v$ uncovered, $A(G)$ is the neighbour set of $D(G)$, and $C(G)$ is the set of all remaining vertices.

**Theorem 3** The set $U = A(G)$ gives equality in the Tutte-Berge Formula. The set $D(G)$ contains all vertices in odd components of $G - U$, and $C(G)$ contains all vertices in even components of $G - U$.

Let $G[D(G)]$ be the subgraph of $G$ induced by $D(G)$. It turns out that every connected component $H$ of $G[D(G)]$ is factor critical, meaning that $H - v$ has a perfect matching for every $v \in V(H)$. Thus for any odd component in $G[D(G)]$ we can actually choose any particular vertex to be left uncovered.

The Edmonds-Gallai Decomposition of a graph can be found as a byproduct of Edmonds’ algorithm for finding a maximum matching. Before describing this algorithm, we need some more basic results. Let $M$ be a matching in a graph $G$. An alternating path (relative to $M$) is a path $P$ whose edges are alternately in $M$ and not in $M$. An augmenting path for $M$ is an alternating path with both endvertices uncovered by $M$. Let $M'$ be the matching obtained by switching $M$-edges and non-$M$-edges along path $P$ (i.e., $M' = M \triangle E(P)$). Then $|M'| = |M| + 1$, which explains why $P$ is called an augmenting path.

**Theorem 4 (Berge)** $M$ is a maximum matching if and only if $G$ contains no $M$-augmenting path.

**Proof:** The “only if” direction is trivial, since any augmenting path can be used to increase the size of $M$. To prove the other direction, suppose that $M$ is not maximum and let $N$ be a maximum matching chosen with minimum symmetric difference with $M$. Consider the subgraph spanned by
Each vertex has degree at most 2, so the subgraph is a disjoint union of paths and cycles. There are no cycles or paths with equal number of edges from $N$ and $M$, since $N \triangle M$ is minimum. There are no paths with more $N$-edges than $M$-edges otherwise $N$ would not be maximum. It follows that every component is an augmenting path for $M$.

Theorem 4 implies the following approach for finding a maximum matching: start with an empty matching and repeatedly find augmenting paths to increase its size. **Edmonds’ Algorithm** uses this approach and gives a specific method for finding augmenting paths. Consider a graph $G = (V, E)$ and a matching $M$ in $G$. Let $X$ be the set of uncovered vertices in $G$. To find an augmenting path for $M$, it will be helpful to define an auxiliary directed graph $G'$ with vertex set $V$ and arc set $A = \{uv \mid \exists x \in V \text{ such that } ux \in E \text{ and } xv \in M\}$. Observe that a directed path in $G'$ corresponds to an (even length) alternating path in $G$. Furthermore, if there is an augmenting path for $M$ then there is a directed path in $G'$ starting at a vertex in $X$ and ending at a neighbour of $X$. Unfortunately, the converse does not necessarily hold: $G$ may contain a directed path in $G'$ starting at a vertex in $X$ and ending at a neighbour of $X$ that does not correspond to an augmenting path. Such a path must necessarily have a prefix that is a *flower*, as shown in this figure.

The dotted arcs show a directed path in the auxiliary graph that starts at a vertex in $X$ and ends at a neighbour of set $X$ but does not correspond to an augmenting path. The graph contains flower, which consists of a *stem* and a *blossom*. The stem is simply an alternating path and the blossom is an odd-length cycle.