18.434 Summary #2

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1 Energy Dissipation in Electrical Network and Thomson's Principle

In the following, we always work with an undirected graph with vertices s, t connected to an external voltage. We explore how the notion of energy can be made precise mathematically and see one application of it.

We propose an alternate characterization of the current function in terms of energy dissipation minimization. This is formalized by Thomson's Principle (stated as Theorem 2 below). We then use it to derive the Rayleigh's Principle, which is difficult to prove without using this alternate characterization.

1.1 Flow

In an undirected graph $G(V, E)$ with vertices s, t, a $s - t$ flow $f : V \times V \longrightarrow \mathbb{R}$ is a function satisfying the following:

- $f_{uv} = 0$ for all $(u, v) \notin E$
- (skew symmetry) $f_{uv} = -f_{vu}$ for all $(u, v) \in V \times V$
- (conservation) $f_u := \sum_v f_{uv} = 0$ for all $u \in V \backslash \{s, t\}$

It is easy to check that the current function is a flow. Furthmore, observe that if f_1 and f_2 are flows, so is $f_1 \pm f_2$.

Theorem 1: For any $s - t$ flow f and $g : V \longrightarrow \mathbb{R}$, $(g(t) - g(s))f_s = \frac{1}{2} \sum_{u,v \in V} (g(v) - g(u)) f_{uv}$. (Think of g as potential)

Observe that this is consistent with the law of conservation of energy when f is the current and g is the potential. In this case, $(g(t) - g(s))f_s$ is the energy dissipated in the entire network and $\sum_{u,v \in V}(g(v) - g(u)) f_{uv}$ is the sum of the energy dissipated in each individual edge.

1.2 Energy

1. The energy dissipated when a current/flow of f_{uv} flows through (u, v) is $f_{uv}^2 R_{uv}$. Total energy dissipation $= \frac{1}{2} \sum_{u,v \in V} f_{uv}^2 R_{uv}$.
2. (For electrical network) Effective resistance: $R_{eff} = (v(t) - v(s))/i_s$. When $i_s = 1$, $R_{eff} = v(t) - v(s)$ and the total energy dissipation is simply R_{eff}.

Theorem 2 (Thomson Principle): Among all unit flows from s to t (i.e. $i_s = 1$), the current is the unique one that minimizes energy dissipation.

Remark: An alternative approach is to use Lagrange Multiplier.

2 Application to Monotonicity

Theorem 3 (Rayleigh Prinicple): The effective resistance R_{eff} of a $s-t$ network is a non-decreasing function of the edge resistances r_e.

Proof: Use Thomson Principle.

Remark 1: Proving it from scratch is highly non-trivial! There are also a few other proofs, some of which can be found in "P.G. Doyle and J.L. Snell, Random Walks and Electric Networks".

Remark 2: When $r_e \longrightarrow \infty$, the edge e is essentially removed from the network; when $r_e \longrightarrow 0$, we "short-circuit" G at $e = (u, v)$ and this is essentially equivalent to contracting u and v.