Solutions to problem set 4

4-2 Notice that one can begin by checking whether \( w_i > w_n + g' \), where \( g' = \sum_{i=1}^{n-1} g_{ni} \), for some \( i < n \). Obviously if this was the case, team \( n \) could not win. Therefore, let us assume \( w_i \leq w_n + \sum_{i=1}^{n-1} g_{ni} \) for all \( i < n \). Moreover, if there was an outcome where team \( n \) won, then it could only get better if it won all of its games, so let us assume that all \( g_{ni} \) games to be played between teams \( n \) and \( i \) are won by team \( n \). If we let \( x_{ij} \) to be the number of games between \( i \) and \( j \) won by \( i \), then team \( n \) has a chance of winning iff there are positive integers \( x_{ij} \) with \( x_{ij} + x_{ji} = g_{ij} = g_{ji}, \ i \neq j \), and \( w_i + \sum_{j=1}^{n-1} x_{ij} \leq w_n + g' \) for all \( i < n \).

Consider the graph \( G \) on \( V = \{s, t\} \cup \{v(i, j)\}_{1 \leq i < j \leq n-1} \cup \{w(i)\}_{1 \leq i \leq n-1} \) with edges classified in these categories (all lower capacities are \( l(e) = 0 \)).

- All edges \( e \) from \( s \) to \( v(i, j) \) with \( u(e) = g_{ij} \).
- All edges \( e_1, e_2 \) from \( v(i, j) \) to \( w(i) \) and to \( w(j) \) with \( u(e_1) = u(e_2) = \infty \).
- All edges \( e \) from \( w(i) \) to \( t \) with \( u(e) = w_n + g' - w_i \).

We claim that team \( n \) can win iff the maximum flow from \( s \) to \( t \) is \( \sum_{i<j} g_{ij} \). Indeed a maximum flow with that value exists iff there exists an integer flow \( x : E \to \mathbb{Z} \) with that flow value exists (since all capacities are integers). If we let \( y_{ij} = x(v(i, j), w(i)) \), such integer flow satisfies \( y_{ij} \geq 0 \), \( g_{ij} = y_{ij} + y_{ji} \) and \( \sum_{j=1}^{n-1} x_{ij} \leq w_n + g' - w_i \). As asserted above, this is equivalent to team \( n \) having some chance.

4-3 Let us construct a digraph \( D = (V', A) \) as following. We create vertices \( w_e \) corresponding to each edge \( e \in E(G) \), source \( s \), and sink \( t \). The set of vertices \( V' \) of \( D \) is

\[
V' := \{s, t\} \cup \{w_e \mid e \in E(G)\} \cup \{v \mid v \in V(G)\}.
\]

For arcs, we let

\[
A := \{(s, w_e) \mid e \in E(G)\} \cup \{(v, t) \mid v \in V(G)\} \cup \{(w_e, v) \mid e \in E(G) \text{ and } v \text{ is an endpoint of } e\}.
\]

Furthermore, we let capacity be \( c(a) = 1 \) if \( a \) is \((s, w_e)\), \( c(a) = +\infty \) if \((w_e, v)\), and \( c(a) = p(v) \) if \( a = (v, t) \).

Let \( e = uv \) be an edge of \( G \). If a unit flow goes through \((s, w_e)\) in a maximum flow, exactly one of \((w_e, u)\) or \((w_e, v)\) will have a unit flow (by integrality). Each corresponds to orienting \( e \) as \((v, u)\) or \((u, v)\), respectively. Moreover, if we push a unit flow through \((s, w_e)\) for every \( e \), then the value of flow going through \((v, t)\) is equal to the indegree of \( v \) in the corresponding orientation. Hence, the problem is equivalent to that the maximum \( s-t \) flow of \( D \) has value \(|E(G)|\).
If the graph cannot be oriented with indegree requirement, then by max-flow min-cut theorem, there is a set $U \subset V'$ such that (1) $s \in U$ and $t \notin U$, and (2) $\sum_{a \in \delta^+(U)} c(a) < |E(G)|$.

Let $S = U \cap V(G)$. We may assume that $U$ contains $w_e$ if and only if $e \subseteq S$, since $c((w_{uv}, v)) = c((w_{uv}, u)) = \infty$. So,

$$|E(G)| > \sum_{a \in \delta^+(U)} c(a) = \sum_{v \in S} p(v) + \sum_{e \notin E(S)} 1.$$ 

We have $\sum_{v \in S} p(v) < |E(S)|$.

4-4 Let $G$ be an undirected graph with minimum degree $\delta(G) \geq k$. Consider the maximum adjacency ordering seen in class:

Choose any vertex $v_1 \in V$ and let $S = \{v_1\}$. Iteratively, find $v_i = \arg \max_{v \in V \setminus S} c(S, \{v\})$ and let $S = S \cup \{v_i\}$. We get an ordering of the vertices $v_1, v_2, \ldots, v_n$.

As shown in the lecture notes, $\{v_n\}$ induces a minimum $(v_{n-1}, v_n)$-cut; the value of such a minimum cut equals to the outdegree of $v_n$, which is at least $k$ by assumption. From duality, the maximum flow value between $v_{n-1}$ and $v_n$ is at least $k$, as desired.