Quiz 1. 

Your Name:

You can have one single-sided handwritten sheet of paper with anything you want on it. You can write your solutions on this exam.

1. Consider the following flow from $s$ to $t$ in the directed graph $G$ below. There are 2 numbers next to each arc $e$: the first is the flow $x_e$ and the second is the capacity $u_e$ (the lower capacities are all 0). So the flow shown is of value 11.

(a) Is the above flow of maximum value? If yes, give a convincing argument and if not, augment it and give a flow of maximum value. (The sketches below are there to help; use them or not, as you prefer.)

(b) Give an $s$-$t$ cut of minimum capacity.

(a) We consider the residual network $G_x$ and find that there is an augmenting path with residual capacity 3. Augmenting the flow along that path we obtain the following flow:

So the original flow was not maximum. New $x$:

This flow has value 14.
Constructing again the residual network, we see that it does not contain any directed path from $s$ to $t$, so this new flow is maximum.

(b) We can get a minimum $s$-$t$ cut by taking the residual network corresponding to this maximum flow and looking at the vertices reachable from $s$ by a directed path.

These vertices are circled here:

and the corresponding cut has value $6 + 4 + 3 + 1 = 14$. This is the minimum $s$-$t$ cut. This matches the value of the maximum flow.
2. Consider the matching $M$ represented by four dashed lines in the following graph:

(a) Identify a flower/blossom with respect to $M$, or state that there are no flowers with respect to $M$.
(b) Is $M$ of maximum size? Give a short justification.
(c) There are many maximum matchings in this graph. Can you find an edge $(i, j)$ which is not in any maximum matching?

(a) The only two exposed vertices are 1 and 5, so a blossom should start at one of these. There is no blossom starting at 5 but there is one starting at 1:

(b) If we contract the above blossom, we get

The matching $M/B$ is maximum since the removal of $U=\{3, 8\}$ gives 4 odd connected components

$$|M/B| = \frac{1}{2} \left( \frac{|V/B|}{6} + \frac{|U|}{2} - o(G/B \cup U) \right) = 2$$

(c) $(i, j) = (3, 8)$ is not in any maximum matching because it is between vertices in $U$, for an optimum set $U$ of the Tutte-Berge formula.
3. Consider the following polyhedron $P$:

$$
P = \{ x \in \mathbb{R}^4 : \begin{align*}
x_1 + x_3 + 2x_4 & \leq 5 \\
x_2 - x_3 + x_4 & \leq 5 \\
x_1 - x_2 & \leq 5 \\
-2x_1 + x_3 - x_4 & \leq 5 \\
x_2 & \leq 0 \\
\end{align*} \} \begin{array}{l}
I_+ \\
I_0 \\
I_- \\
\end{array}
$$

Consider its projection $P' = \{ (x_1, x_2, x_3) : \exists x_4 \text{ with } (x_1, x_2, x_3, x_4) \in P \}$. Express $P'$ in terms of linear inequalities.

We use Fourier-Motzkin elimination.

We'll get $|I_+| \cdot |I_-| + |I_0| = 2 \cdot 2 + 1 = 5$ inequalities

$$
\begin{align*}
\begin{cases}
x_1 + x_3 + 2x_4 & \leq 5 \\
-2x_1 + x_3 - x_4 & \leq 5
\end{cases} & \Rightarrow -3x_1 + 3x_3 \leq 15 \\
x_1 + x_3 + 2x_4 & \leq 5 \\
x_2 - 2x_4 & \leq 5
\end{cases} \Rightarrow \begin{cases}
x_1 + x_2 + x_3 & \leq 10
\end{cases}
$$

$$
\begin{cases}
x_2 - x_3 + x_4 & \leq 5 \\
-2x_1 + x_3 - x_4 & \leq 5
\end{cases} \Rightarrow -2x_1 + x_2 \leq 10
$$

$$
\begin{cases}
x_2 - x_3 + x_4 & \leq 5 \\
x_2 & \leq 5
\end{cases} \Rightarrow \begin{cases}
3x_2 - 2x_3 \leq 15
\end{cases}
$$

So, $P' = \left\{ x \in \mathbb{R}^3 : \begin{align*}
-3x_1 + 3x_3 & \leq 15 \\
x_1 + x_2 + x_3 & \leq 10 \\
-2x_1 + x_2 & \leq 10 \\
3x_2 - 2x_3 & \leq 15 \\
x_1 - x_2 & \leq 5
\end{align*} \right\}$ from $I_0$
4. Suppose an employer is trying to decide which tasks to do for the next \(m\) days. He knows the number \(b_i\) of employees available on day \(i\) for \(i = 1, \ldots, m\). There are \(n\) possible tasks, and task \(j\) starts (at the beginning of) day \(s_j\) and ends (at the end of) day \(t_j\). Each task requires one employee (but not necessarily the same employee) for all days from \(s_j\) up to and including \(t_j\). The employer wants to select the maximum number of tasks that can be done by his employees.

Let \(x_j = 1\) if task \(j\) is selected and 0 otherwise, and consider the linear program:

\[
\text{Max} \sum_{j=1}^{n} x_j \tag{1}
\]

\[
s.t. \sum_{j : s_j \leq i \leq t_j} x_j \leq b_i \quad i = 1, \ldots, m \tag{2}
\]

\[
x_j \geq 0 \quad j = 1, \ldots, n. \tag{3}
\]

(a) Let the matrix \(A\) correspond to the set of constraints above (i.e. (2) is \(Ax \leq b\)). Is \(A\) totally unimodular (TU)? Either prove or disprove.

(b) What does this imply for the above linear program?

(a) Yes, the matrix \(A\) is TU

\[A\] is \(m \times n\) and column \(j\) has 1's in a consecutive set of rows from \(s_j\) to \(t_j\). Any such matrix \(A\) with 0-1 entries with all 1's consecutive in every column is TU. Indeed given any subset \(R\) of rows, we can alternate (between) putting the rows between \(R_1\) and \(R_2\), and the resulting partition will satisfy:

\[
\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{-1, 0, 1\} \quad \forall j
\]

(it will be 1 if \(R_1\) starts with a 1 in column \(j\) and ends with a 0)

(b) It means that for any choice of integral \(b_i\)'s, the above linear program has an integral optimum solution.
5. Consider the permutahedron \( P_n \subseteq \mathbb{R}^n \) of order \( n \) which is defined as the convex hull of all permutations \( \sigma \) of \([n] = \{1, \cdots, n\}\). We saw in class that

\[
P_n = \{ x \in \mathbb{R}^n : \sum_{i \in [n]} x_i = \binom{n+1}{2} \sum_{i \in \emptyset \cup S} (|S|+1) \forall \emptyset \neq S \subset [n] \}\]

Part (c) can be answered independently.

(a) What is the dimension of \( P_n \)? Just give the answer, no need to formally prove it.

(b) Does the inequality (5) for \( S = \{1,2,3,\cdots,k\} \) define a facet of \( P_n \)? Justify. How many facets does \( P_n \) have?

(c) To every permutation \( \sigma \) of \([n]\), we can associate a perfect matching \( M \) in an \( n \times n \) complete bipartite graph: \( M_\sigma = \{(i, \sigma(i)) : 1 \leq i \leq n\} \). Now consider the polytope \( Q = \text{conv}\{x, y \in \mathbb{R}^n \times \mathbb{R}^n : x \) is a permutation \( \sigma \) of \([n]\) and \( y \) is the incidence vector of \( M_\sigma \}\). Observe that \( Q \) has precisely \( n! \) vertices (just one for each permutation \( \sigma \)) and \( Q \) projects onto \( P_n \); \( P_n = \{ x \in \mathbb{R}^n : 3y \in \mathbb{R}^n \) with \( (x, y) \in Q \}\).

Give a complete description of \( Q \) in terms of linear inequalities. Justify. How many facets does \( Q \) have?

\[ \text{(a) } \dim(P_n) = n-1 \quad (n-1 \text{ for the equality}) \]

\[ \text{(b) Yes. The following points satisfy the inequality at equality and are affinely independent:} \]

\[
\begin{align*}
x_i^{(i)} &= i & \text{for } i = 1, \cdots, n & \Rightarrow 1 \text{ point} \\
x_i^{(j)} &= \begin{cases} j & \text{if } i = j \\ 1 & \text{otherwise} \end{cases} & \text{for } j = 2, \cdots, k & \Rightarrow k-1 \text{ points} \\
x_i^{(j)} &= \begin{cases} k+1 & \text{if } i = j \\ j & \text{if } i = k+1 \\ i & \text{otherwise} \end{cases} & \text{for } j = k+2, \cdots, n & \Rightarrow n-k-1 \text{ points} \\
\end{align*}
\]

\[
\text{To show they are affinely independent, assume that } \sum_j a_j x^{(i)} = 0 \text{ and } \sum_j a_j = 0 \]

\[
\sum_j a_j x^{(i)} = 0 \quad \Rightarrow \quad \sum_j a_j (x^{(i)} - i) = 0
\]

\[
\sum_j a_j = 0 \quad \Rightarrow \quad \text{for } i = 2, \cdots, k \text{ and } i = k+2, \cdots, n \text{ this is 0 except if } j = i
\]
This means that \( \alpha_j = 0 \) for \( j = 2, \ldots, k \) and for \( j = k+2, \ldots, n \), and this together with \( \sum_j \alpha_j = 0 \) implies that the last remaining \( \alpha \), namely \( \alpha_n \), is also 0. Thus \( \alpha_j = 0 \) \( \forall j \) showing affinely independence. \( P_n \) has \( 2^{n-2} \) facets \( \Rightarrow S \neq \emptyset \) and \( S \neq [n] \).

(c) 

\[
Q = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^{n^2} : \begin{array}{l}
\sum_{i=1}^{n} y_{ij} = 1 \quad \forall j = 1, \ldots, n \\
\sum_{j=1}^{n} y_{ij} = 1 \quad \forall i = 1, \ldots, n \\
y_{ij} \geq 0 \quad \forall i, j
\end{array} \right\}
\]

(1)-(2)-(3) is the convex hull of all perfect matchings \( y \), and integrality is preserved when I add (4) since for any integer \( y \), the reality \( x \) given by the inequalities (4) is integral.

The number of facets of \( Q \) is \( n^2 \) given by inequalities (3).

(This is much less than the \( 2^{n-2} \) of \( P_n \)).